

LIMIT MODELS IN CLASSES WITH AMALGAMATION

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ABSTRACT. In abstract elementary classes limit models are sometimes the appropriate substitute for saturated models. For Galois-stable abstract elementary classes which satisfy the amalgamation property, we prove under the assumption that there is a mildly behaved dependence relation, that for any model M , any two limit models over M of the same cardinality are isomorphic. This is useful in dealing with Shelah's categoricity conjecture.

1. INTRODUCTION

In 1977, Shelah, building on the work of Jónsson and Fraïssé, identified a non-elementary context in which a model theoretic analysis could be carried out. Shelah began to study classes of models equipped with a partial order which exhibit many of the properties that the models of a first order theory have with respect to the elementary submodel relation. Such classes were named abstract elementary classes. They are broad enough to generalize $L_{\omega_1, \omega}(\mathbf{Q})$. We reproduce the definition here.

Definition 1.1. Let \mathcal{K} be a class of structures all in the same similarity type $L(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on \mathcal{K} . The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class, AEC for short* iff

- A0 (Closure under isomorphism)
 - (a) For every $M \in \mathcal{K}$ and every $L(\mathcal{K})$ -structure N if $M \cong N$ then $N \in \mathcal{K}$.
 - (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.
- A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
- A2 Let M, N, M^* be $L(\mathcal{K})$ -structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.
- A3 (Downward Löwenheim-Skolem) There exists a cardinal $\text{LS}(\mathcal{K}) \geq \aleph_0 + |L(\mathcal{K})|$ such that for every

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$M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + \text{LS}(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

- (a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$ -increasing (i.e. $i < j \implies M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i < \mu} M_i \in \mathcal{K}$ and $\bigcup_{i < \mu} M_i \prec_{\mathcal{K}} N$.
- (b) For every regular μ , if $\{M_i : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$ -increasing then $\bigcup_{i < \mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i < \mu} M_i$.

For M and $N \in \mathcal{K}$ a monomorphism $f : M \rightarrow N$ is called an \mathcal{K} -embedding iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to “ id_M is a \mathcal{K} -embedding from M into N ”.

For $M_0 \prec_{\mathcal{K}} M_1$ and $N \in \mathcal{K}$, the formula $f : M_1 \xrightarrow{M_0} N$ stands for f is a \mathcal{K} -embedding such that $f \upharpoonright M_0 = \text{id}_{M_0}$.

For a class \mathcal{K} and a cardinal $\mu \geq \text{LS}(\mathcal{K})$ let

$$\mathcal{K}_\mu := \{M \in \mathcal{K} : \|M\| = \mu\}.$$

In reality, abstract elementary classes were not as approachable as one would expect and much work in non-elementary model theory takes place in contexts which additionally satisfy the amalgamation property:

Definition 1.2. Let $\mu \geq \text{LS}(\mathcal{K})$. We say that \mathcal{K} has the μ -amalgamation property (μ -AP) iff for any $M_\ell \in \mathcal{K}_\mu$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there are $N \in \mathcal{K}_\mu$ and \mathcal{K} -embeddings $f_\ell : M_\ell \rightarrow N$ such that $f_\ell \upharpoonright M_0 = \text{id}_{M_0}$ for $\ell = 1, 2$.

A model $M_0 \in \mathcal{K}_\mu$ satisfying the above requirement is called an *amalgamation base*.

We say that \mathcal{K} has the *amalgamation property* (AP) iff any triple of models from $\mathcal{K}_{\geq \text{LS}(\mathcal{K})}$ can be amalgamated.

Remark 1.3. (1) Using the isomorphism axioms we can see that \mathcal{K} has the λ -AP iff for any $M_\ell \in \mathcal{K}_\lambda$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell \in \{1, 2\}$) there are $N \in \mathcal{K}_\lambda$ and $f : M_1 \xrightarrow{M_0} N$ such that

$$N \succ_{\mathcal{K}} M_2.$$

- (2) Using the axioms of AECs it is not difficult to prove that if \mathcal{K} has the λ -AP for every $\lambda \geq \text{LS}(\mathcal{K})$ then \mathcal{K} has the AP.

A stronger version of the amalgamation property is

Definition 1.4. Let \mathcal{K} be an abstract class. \mathcal{K} has the λ -Disjoint Amalgamation Property iff for every $M_\ell \in \mathcal{K}_\lambda$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there are $N \in \mathcal{K}_\lambda$ which is a \mathcal{K} -extension of M_2 and a \mathcal{K} -embedding $f : M_1 \xrightarrow{M_0} N$ such that $f[M_1] \cap M_2 = M_0$.

We say that a class has the *disjoint amalgamation property* iff it has the λ -disjoint amalgamation property for every $\lambda \geq \text{LS}(\mathcal{K}) + \aleph_0$. We write DAP for short.

An application of the compactness theorem establishes:

Fact 1.5. *If T is a complete first-order theory then $\langle \text{Mod}(T), \prec \rangle$ has the λ -DAP for all $\lambda \geq |L(T)| + \aleph_0$*

The roots of the following fact can be traced back to Jónsson's 1960 paper [Jo], the present formulation is from [Gr1]:

Fact 1.6. *Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC and $\lambda \geq \kappa > \text{LS}(\mathcal{K})$ such that $K_{<\lambda}$ has the AP. Suppose $M \in \mathcal{K}$.*

If $\lambda^{<\kappa} = \lambda \geq \|M\|$ then there exists $N \succ M$ of cardinality λ which is κ -model-homogeneous.

Thus if an AEC \mathcal{K} has the amalgamation property then like in first-order stability theory we may assume that there is a large model-homogeneous $\mathfrak{C} \in \mathcal{K}$, that acts like a monster model.

We will refer to the model \mathfrak{C} as the *monster model*. All models considered will be of size less than $|\mathfrak{C}|$, and we will find realizations of types inside this monster model.

From now on, we assume that the monster model \mathfrak{C} has been fixed.

The notion of type as a set of formulas, does not seem to be as nicely behaved as in first-order logic. Thus we need a replacement which was introduced by Shelah in [Sh 394], in order to avoid confusion with the classical notion, following [Gr2] we call this new notion *Galois-type*.

Since in this paper we deal only with AECs with the amalgamation property, the notion of Galois type has a simpler definition than in the general case.

Definition 1.7 (Galois types). Suppose that \mathcal{K} has the AP.

- (1) Given $M \in \mathcal{K}$ consider the action of $\text{Aut}_M(\mathfrak{C})$ on \mathfrak{C} , for an element $a \in |\mathfrak{C}|$ let $\text{ga-tp}(a/M)$ denote the *Galois type of a over M* which is defined as the orbit of a under $\text{Aut}_M(\mathfrak{C})$.
- (2) For $M \in \mathcal{K}$, we let

$$\text{ga-S}(M) = \{ \text{ga-tp}(a/M) : a \in |\mathfrak{C}| \}.$$

- (3) \mathcal{K} is λ -Galois-stable iff

$$N \in \mathcal{K}_\lambda \implies |\text{ga-S}(N)| \leq \lambda.$$

- (4) Given $p \in \text{ga-S}(M)$ and $N \in \mathcal{K}$, we say that p is *realized* by $a \in N$, iff $\text{ga-tp}(a/M) = p$.

Like in the first-order case we will write $a \models p$ when a is a realization of p .

For a more detailed discussion of Galois types, extensions and restrictions, equivalent and more general formulations, the reader may consult [Gr2].

The main concept of this paper is Shelah's *limit model* which is used as a substitute for saturation. Before defining these, we first need to define universal extensions as they are the building blocks of limit models:

Definition 1.8. (1) Let κ be a cardinal $\geq \text{LS}(\mathcal{K})$. We say M^* is κ -universal over N iff for every $N' \in \mathcal{K}_\kappa$ with $N \prec_{\mathcal{K}} N'$ there exists a \mathcal{K} -embedding $g : N' \xrightarrow[N]{\text{id}} M^*$ such that:

$$\begin{array}{ccc} & N' & \\ & \uparrow \text{id} & \searrow g \\ N & \xrightarrow{\text{id}} & M^* \end{array}$$

(2) We say M^* is *universal over N* or M^* is a *universal extension of N* iff M^* is $\|N\|$ -universal over N .

Theorem 1.9 (Existence). *Let \mathcal{K} be an AEC without maximal models and suppose it is Galois-stable in μ . If \mathcal{K} has the amalgamation property then for every $N \in \mathcal{K}_\mu$ there exists $M^* \not\prec_{\mathcal{K}} N$, universal over N of cardinality μ .*

This theorem was stated without proof in [Sh 600], for a proof see [GrVa1] or [Gr1].

In [KoSh] and in [Sh 576] Shelah introduced a substitute for saturated models under the name of (μ, α) -saturated models. As Shelah himself in later papers refers to these models as *limit models* we use the more recent terminology.

Definition 1.10. [Limit models] Let $\mu \geq \text{LS}(\mathcal{K})$ and $\alpha < \mu^+$ a limit ordinal and $N \in \mathcal{K}_\mu$. We say that M is (μ, α) -limit over N iff there exists an increasing and continuous chain $\{M_i \mid i < \alpha\} \subseteq \mathcal{K}_\mu$ such that $M_0 = N$, $M = \bigcup_{i < \alpha} M_i$ and M_{i+1} is universal over M_i for all $i < \alpha$.

From Theorem 1.9 we get that for $\alpha < \mu^+$ there always exists a (μ, α) -limit model provided \mathcal{K} has the AP, has no maximal models and is μ -Galois-stable.

The following theorem partially clarifies the analogy with saturated models:

Theorem 1.11. *Let T be a complete first-order theory and let \mathcal{K} be the elementary class $\text{Mod}(T)$ with the usual notion of elementary submodels.*

- (1) *Suppose T is superstable. If M is (μ, δ) -limit model for δ a limit ordinal, then M is saturated.*
- (2) *Suppose T is stable. If M is (μ, δ) -limit model for δ a limit ordinal with $\delta \geq |T|^+$, then M is saturated.*

Thus under mild model-theoretic assumptions in elementary classes limit models are unique. This raises the following natural question for the situation in AECs:

Question 1.12 (Uniqueness problem). *Let \mathcal{K} be an AEC, $\mu \geq \text{LS}(\mathcal{K})$, σ_1, σ_2 limit ordinals $< \mu^+$ and $M \in \mathcal{K}_\mu$. Suppose that N_ℓ is (μ, σ_ℓ) -limit models over M for $\ell = 1, 2$. What “reasonable” assumptions on \mathcal{K} will imply that $\exists f : N_1 \cong_M N_2$?*

Using back and forth arguments one can show that when $\sigma_1 = \sigma_2$ then we get uniqueness without any assumptions on \mathcal{K} , in fact $\text{cf } \sigma_1 = \text{cf } \sigma_2$ suffices. More precisely:

Fact 1.13. *Let $\mu \geq \text{LS}(\mathcal{K})$ and $\sigma < \mu^+$. If M_1 and M_2 are (μ, σ) -limits over M , then there exists an isomorphism $g : M_1 \rightarrow M_2$ such that $g \upharpoonright M = \text{id}_M$. Moreover if M_1 is a (μ, σ) -limit over M_0 ; N_1 is a (μ, σ) -limit over N_0 and $g : M_0 \cong N_0$, then there exists a $\prec_{\mathcal{K}}$ -mapping, \hat{g} , extending g such that $\hat{g} : M_1 \cong N_1$.*

Fact 1.14. *Let μ be a cardinal and σ a limit ordinal with $\sigma < \mu^+ \leq \lambda$. If M is a (μ, σ) -limit model, then M is a $(\mu, \text{cf}(\sigma))$ -limit model.*

Thus Question 1.12 is meaningful for the case where $\text{cf } \sigma_1 \neq \text{cf } \sigma_2$. The main result of this paper is:

Theorem 1.15 (Main Theorem). *Let \mathcal{K} be an AEC and $\mu > \text{LS}(\mathcal{K})$. Suppose \mathcal{K} is satisfying the μ -DAP. If \mathcal{K} is μ -Galois-stable, does not have long splitting chains, and satisfies locality of splitting¹, then any two (μ, σ_ℓ) -limits over M for $(\ell \in \{1, 2\})$ are isomorphic over M .*

Notice that μ -DAP is occasionally a property we get for free if the class \mathcal{K} has an axiomatization in a logic with sufficient compactness; essentially, Robinson's consistency property is enough. In other occasions DAP is a known corollary of categoricity, even when AP is not assumed (see [ShVi] and [Va]).

Approximations to Theorem 1.15 and its relatives were considered by several authors:

Kolman and Shelah in [KoSh] prove the uniqueness of limit models of cardinality μ in λ -categorical AECs that are axiomatized by a $L_{\kappa, \omega}$ -sentence where $\lambda > \mu$ and κ is a measurable cardinal. Both the measurability of κ and the categoricity are used integrally in their proof.

Shelah in Theorem 6.3 of [Sh 394] gets uniqueness of limit models of cardinality μ for classes with the amalgamation property under little more than categoricity in some $\lambda > \mu > \text{LS}(\mathcal{K})$ together with existence of arbitrarily large models. An analysis of Ehrenfeucht-Mostowski models possible only when there are categoricity assumptions, gives uniqueness. Our assumptions in this paper may all be derived from the assumptions in [Sh 394], but our arguments do not require the Ehrenfeucht-Mostowski machinery.

Shelah in [Sh 576] (see Claim 7.8) proved a special case of the uniqueness of limit models under the assumption of μ -AP, categoricity in μ and in μ^+ as well as assuming $K_{\mu^{++}} \neq \emptyset$. In that paper Shelah needs to produce *reduced types* and use some of their special properties.

[ShVi] attempted to prove a uniqueness theorem without assuming any form of amalgamation; however, they assumed that \mathcal{K} is categorical in some $\lambda > \text{Hanf}(\mathcal{K}) + \mu$ and that every model in \mathcal{K} has a proper extension.

¹See Assumption 2.4 for the precise description of long splitting chains and locality

VanDieren in [Va] managed to prove the above uniqueness statement under the assumptions of [ShVi] together with the additional assumption that $\mathcal{K}_\mu^{am} := \{M \in \mathcal{K}_\mu \mid M \text{ is an amalgamation base}\}$ is closed under unions of increasing $\prec_{\mathcal{K}}$ chains.

Finally, the uniqueness of limit models appears as a requirement in the definition of good frames (Part (B) of the definition of good frames, Definition 2.1 of [Sh 600]) and of λ frames (Definition 0.2 of [Sh 705]) where Shelah attempts to axiomatize superstability in AECs.

We are particularly interested in Theorem 1.15 not only for the sake of generalizing Shelah's result from [Sh 576] but due to the fact that the first and second author use this uniqueness theorem in a crucial step to prove:

Theorem 1.16 (Upward categoricity theorem). *Suppose that \mathcal{K} has arbitrarily large models, is χ -tame and satisfies the amalgamation and joint embedding properties. Let λ be such that $\lambda > \text{LS}(\mathcal{K})$ and $\lambda \geq \chi$. If \mathcal{K} is categorical in λ^+ then \mathcal{K} is categorical in all $\mu \geq \lambda^+$.*

The proof of this categoricity theorem is in [GrVa2].

2. THE SETTING

For the remainder of the paper we assume that \mathcal{K} is an AEC satisfying both the μ -amalgamation property and the μ -disjoint amalgamation property. We will prove the uniqueness of limit models in classes which are equipped with a moderately well-behaved dependence relation. Thus we will additionally assume that \mathcal{K} is stable in μ . We will use μ -splitting as the dependence relation, but any dependence relation which is local and has existence, uniqueness and extension properties suffices.

Definition 2.1. A type $p \in \text{ga-S}(M)$ μ -splits over N if and only if N is a $\prec_{\mathcal{K}}$ -submodel of M of cardinality μ and there there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a \mathcal{K} -mapping h such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$ and $h : N_1 \rightarrow N_2$ with $h \upharpoonright N = \text{id}_N$ and $p \upharpoonright N_2 \neq h(p \upharpoonright N_1)$.

The existence property for non- μ -splitting types follows from Galois stability in μ :

Fact 2.2 (Claim 3.3 of [Sh 394]). *Assume \mathcal{K} is an abstract elementary class and is Galois-stable in μ . For every $M \in \mathcal{K}_{\geq \mu}$ and $p \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_\mu$ such that p does not μ -split over N .*

The uniqueness and extension property of non- μ -splitting types holds for types over limit models:

Fact 2.3 (Theorem I.4.15 of [Va]). *Suppose that \mathcal{K} is an AEC. Let $N, M, M' \in \mathcal{K}_\mu$ be such that M' is universal over M and M is universal over N . If $p \in \text{ga-S}(M)$ does not μ -split over N , then there is a unique $p' \in \text{ga-S}(M')$ such that p' extends p and p' does not μ -split over N .*

Here are the assumptions of the paper:

- Assumption 2.4.** (1) \mathcal{K} is an AEC with the μ -amalgamation property.
(2) \mathcal{K} satisfies the μ -disjoint amalgamation property.
(3) \mathcal{K} is stable in μ .
(4) μ -splitting in \mathcal{K} satisfies the following locality and existence properties.

For every sequence $\langle M_i \mid i < \alpha \rangle$ of limit models of cardinality μ and for every $p \in \text{ga-S}(M_\alpha)$ we have that

- (a) If for every $i < \alpha$, $p \upharpoonright M_i$ does not μ -split over M_0 , then p does not μ -split over M_0 .
(b) There exists $i < \alpha$ such that p does not μ -split over M_i .

Remark 2.5. Categoricity in a cardinal $\lambda > \mu$ implies all parts of Assumption 2.4 in classes satisfying AP [Sh 394] if one assumes that there are models of arbitrarily large cardinalities.

The Disjoint Amalgamation Property (DAP) comes for free in First Order Contexts, in Homogeneous Classes and in Local AECs. It also holds for cats consisting of existentially closed models of positive Robinson theories ([Za]). In each of these contexts dependence relations satisfying Assumption 2.4 have been developed. Finally, the locality and existence of non- μ -splitting extensions are akin to consequences of superstability in first order logic.

3. STRONG TYPES

Under the assumption of μ -stability, we can define *strong types* as in [ShVi]. These strong types will allow us to achieve a better control of extensions of towers of models than what we obtain using just Galois types.

Definition 3.1 (Definition 3.2.1 of [ShVi]). For M a (μ, θ) -limit model (see definition 1.10),

- (1) Let

$$\mathfrak{St}(M) := \left\{ (p, N) \left| \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \text{ is a } (\mu, \theta) \text{ - limit model;} \\ M \text{ is universal over } N; \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \text{ does not } \mu \text{ - split over } N. \end{array} \right. \right\}$$

- (2) For types $(p_l, N_l) \in \mathfrak{St}(M)$ ($l = 1, 2$), we say $(p_1, N_1) \sim (p_2, N_2)$ iff for every $M' \in \mathcal{K}_\mu$ extending M there is a $q \in \text{ga-S}(M')$ extending both p_1 and p_2 such that q does not μ -split over N_1 and q does not μ -split over N_2 .
(3) Two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are *parallel* iff for every M' of cardinality μ extending M_1 and M_2 there exists $q \in \text{ga-S}(M')$ such that q extends both p_1 and p_2 and q does not μ -split over N_1 and N_2 .

Notation 3.2. Let $M, M' \in \mathcal{K}_\mu$, Suppose that M is a $\prec_{\mathcal{K}}$ -submodel of M' . For $(p, N) \in \mathfrak{St}(M')$, if M is universal over N , we define the restriction $(p, N) \upharpoonright M \in \mathfrak{St}(M')$ to be $(p \upharpoonright M, N)$.

If we write $(p, N) \upharpoonright M$, we mean that p does not μ -split over N and M is universal over N .

Notice that \sim is an equivalence relation on $\mathfrak{St}(M)$ (see [Va]). Stability in μ implies that there are few strong types over any model of cardinality μ :

Fact 3.3 (Claim 3.2.2 (3) of [ShVi]). *If \mathcal{K} is Galois-stable in μ , then for any $M \in \mathcal{K}$ of cardinality μ , $|\mathfrak{St}(M)/\sim| \leq \mu$.*

4. TOWERS

To each (μ, θ) -limit model M we can naturally associate a continuous tower $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \theta \rangle$ witnessing that M is a (μ, θ) -limit model (that is, $\bigcup_{i < \theta} M_i = M$ and M_{i+1} is universal over M_i). Further, by Facts 1.13 and 1.14 we can require that this tower satisfy additional requirements such as M_{i+1} is a limit model over M_i .

To prove the uniqueness of limit models we will construct a model which is simultaneously a (μ, θ_1) -limit model over some fixed model M and a (μ, θ_2) -limit model over M . Notice that, by Fact 1.13, it is enough to construct a model M^* that is simultaneously a (μ, ω) -limit model and a (μ, θ) -limit model for arbitrary θ . By Fact 1.14 we may assume that θ is a limit ordinal $< \mu^+$ such that $\theta = \mu \cdot \theta$.

So, we actually construct an array of models with $\omega + 1$ rows, such that the bottom corner of the array (M^*) will be a (μ, θ) -limit model witnessed by a tower of models as described in the first paragraph of this section. This tower will appear in the last column of the array. We will see that M^* is a (μ, θ) -limit model by examining the last (the ω th) row of the array. This last row will be an $\prec_{\mathcal{K}}$ -increasing sequence of models, \bar{M}^* of length θ_2 . However we will not be able to guarantee that M_{i+1}^* is universal over M_i^* . Thus we need another method to conclude that M^* is a (μ, θ_2) -limit model. This involves attaching more information to our tower \bar{M}^* .

Under the assumption of Galois-stability, given any sequence $\langle a_i \mid i < \theta \rangle$ of elements with $a_i \in M_{i+1} \setminus M_i$, we can identify $N_i \prec_{\mathcal{K}} M_i$ such that $\text{ga-tp}(a_i/M_i)$ does not μ -split over N_i . Furthermore, by Assumption 2.4, we may choose this N_i such that M_i is a limit model over N_i . We abbreviate this situation by a tower $(\bar{M}, \bar{a}, \bar{N})$:

Definition 4.1. We denote by $\mathcal{K}_{\mu, \theta}^*$ the set of towers $(\bar{M}, \bar{a}, \bar{N})$ where $\bar{M} = \langle M_i \mid i < \theta \rangle$ is a $\prec_{\mathcal{K}}$ -increasing sequence of limit models of cardinality μ ; $\bar{a} = \langle a_i \mid i + 1 < \theta \rangle$ and $\bar{N} = \langle N_i \mid i + 1 < \theta \rangle$ satisfy $a_i \in M_{i+1} \setminus M_i$; $\text{ga-tp}(a_i/M_i)$ does not μ -split over N_i ; and M_i is universal over N_i .

Notation 4.2. Similarly define $\mathcal{K}_{\mu, I}^*$ where I is a well-ordered set. We use the notation $i + 1$ for the successor of i in I when it is clear which index set I we are using. At times there may be more than one index set, we will write $\text{succ}_I(i)$ for the successor of i in I to distinguish it from the successor of i in another index set. Finally when I is a sub-order of I' for

any $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ we write $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright I$ for the tower $\mathcal{K}_{\mu, I}^*$ given by $\bar{M} \upharpoonright I = \langle M_i \mid i \in I \rangle$, $\bar{N} \upharpoonright I = \langle N_i \mid i \in I \rangle$ and $\bar{a} \upharpoonright I = \langle a_i \mid i \in I \rangle$.

For a tower $(\bar{M}, \bar{a}, \bar{N})$, it was shown in [ShVi] and [Va], that even if M_{i+1} is not universal over M_i , one can conclude that $\bigcup_{i < \theta} M_i$ is a (μ, θ) -limit model provided $\theta = \mu \cdot \theta$ and for every $i < \theta$, and every strong type (p, N) over M_i , there is $j < i + \mu$ such that $(\text{ga-tp}(a_j/M_i), N_i) \sim (\text{ga-tp}(a_j/M_i), N)$. In fact slightly less is required:

Definition 4.3. Suppose that I is a well-ordered set such that there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta \rangle$ of I of order type θ such that there are $\mu \cdot \omega$ many element between i_α and $i_{\alpha+1}$.

Let $(\bar{M}, \bar{a}, \bar{N})$ be a tower indexed by I such that each M_i is a (μ, σ) -limit model. For each i , let $\langle M_i^\gamma \mid \gamma < \sigma \rangle$ witness that M_i is a (μ, σ) -limit model.

A tower $(\bar{M}, \bar{a}, \bar{N})$ is *full relative to* $(M_i^\gamma)_{\gamma < \sigma, i \in I}$ iff for every $\gamma < \sigma$ and every $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I$ with $i \leq j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j), N_j)$ and (p, M_i^γ) are parallel.

Fact 4.4. Let θ be a limit ordinal $< \mu^+$ satisfying $\theta = \mu \cdot \theta$. Suppose that I is a well-ordered set such that there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta \rangle$ of I of order type θ such that there are $\mu \cdot \omega$ many element between i_α and $i_{\alpha+1}$.

Let $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}$ be a tower made up of (μ, σ) -limit models,. If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}$ is full relative to $(M_i^\gamma)_{i \in I, \gamma < \sigma}$, then $M := \bigcup_{i \in I} M_i$ is a (μ, θ) -limit model.

Proof. Without loss of generality we may assume that \bar{M} is continuous. Let M' be a (μ, θ) -limit model over M_{i_0} witnessed by $\langle M'_\alpha \mid \alpha < \theta \rangle$. By Disjoint Amalgamation, we may assume that $M' \cap M = M_{i_0}$. Since $\theta = \mu \cdot \theta$, we may also arrange things so that the universe of M'_α is $\mu \cdot \alpha$ and $\alpha \in M'_{\alpha+1}$.

We will construct an isomorphism between M and M' by induction on $\alpha < \theta$. Define an increasing and continuous sequence of $\prec_{\mathcal{K}}$ -mappings $\langle h_\alpha \mid \alpha < \theta \rangle$ such that

- (1) $h_\alpha : M_{i_\alpha+j} \rightarrow M'_{\alpha+1}$ for some $j < \mu \cdot \omega$
- (2) $h_0 = \text{id}_{M_{0,0}}$ and
- (3) $\alpha \in \text{rg}(h_{\alpha+1})$.

For $\alpha = 0$ take $h_0 = \text{id}_{M_{0,0}}$. For α a limit ordinal let $h_\alpha = \bigcup_{\beta < \alpha} h_\beta$. Since \bar{M} is continuous, the induction hypothesis gives us that h_α is a $\prec_{\mathcal{K}}$ -mapping from M_{i_α} into M'_α allowing us to satisfy condition (1) of the construction.

Suppose that h_α has been defined. Let $j < \mu \cdot \omega$ be such that $h_\alpha : M_{i_\alpha+j} \rightarrow M'_{\alpha+1}$. There are two cases: either $\alpha \in \text{rg}(h_\alpha)$ or $\alpha \notin \text{rg}(h_\alpha)$. First suppose that $\alpha \in \text{rg}(h_\alpha)$. Since $M'_{\alpha+2}$ is universal over $M'_{\alpha+1}$, it is also universal over $h_\alpha(M_{i_\alpha+j})$. This allows us to extend h_α to $h_{\alpha+1} : M_{i_{\alpha+1}} \rightarrow M'_{\alpha+2}$.

Now consider the case when $\alpha \notin \text{rg}(h_\alpha)$. Since $\langle M_{i_\alpha+j}^\gamma \mid \gamma < \sigma \rangle$ witnesses that $M_{i_\alpha+j}$ is a (μ, σ) -limit model, by Assumption 2.4, there exists $\gamma < \sigma$ such that $\text{ga-tp}(\alpha/M_{i_\alpha+j})$ does not μ -split over $M_{i_\alpha+j}^\gamma$. By our choice of \bar{M}' disjoint from \bar{M} outside of M_{i_0} , we know that $\alpha \notin M_{i_\alpha+j}$. Thus

$\text{ga-tp}(\alpha/M_{i_\alpha+j})$ is non-algebraic. By relative fullness of $(\bar{M}, \bar{a}, \bar{N})$, there exists j' with $j \leq j' < i_{\alpha+1}$ such that $(\text{ga-tp}(\alpha/M_{i_\alpha+j'}), M_{i_\alpha+j}^\gamma)$ is parallel to $(\text{ga-tp}(a_{i_{\alpha+1}+j'}/M_{i_{\alpha+1}+j'}), N_{i_{\alpha+1}+j'})$. In particular we have that

$$(*) \quad \text{ga-tp}(a_{i_{\alpha+1}+j'}/M_{i_\alpha+j}) = \text{ga-tp}(\alpha/M_{i_\alpha+j}).$$

We can extend h_α to an automorphism h' of \mathfrak{C} . An application of h' to $(*)$ gives us

$$(**) \quad \text{ga-tp}(h'(a_{i_{\alpha+1}+j'})/h_\alpha(M_{i_\alpha+j})) = \text{ga-tp}(\alpha/h_\alpha(M_{i_\alpha+j})).$$

Since $M'_{\alpha+2}$ is universal over $h_\alpha(M_{i_\alpha})$, we may extend h_α to a \mathcal{K} -mapping $h_{\alpha+1} : M_{i_{\alpha+1}+j'} \rightarrow M'_{\alpha+2}$ such that $h_{\alpha+1}(a_{i_{\alpha+1}+j'}) = \alpha$.

Let $h := \bigcup_{\alpha < \theta} h_\alpha$. Clearly $h : M \rightarrow M'$. To see that h is an isomorphism, notice that condition (3) of the construction forces h to be surjective. \dashv

5. UNIQUENESS OF LIMIT MODELS

Recall that our goal is to construct a model M^* that is simultaneously a (μ, ω) -limit model and a (μ, θ) -limit model where θ is a limit ordinal $< \mu^+$ such that $\theta = \mu \cdot \theta$.

We now begin the construction of M^* . The goal will be to build a $\theta+1$ by $\omega+1$ array of models so that the bottom row of the array is a relatively full tower. We also need to be able to guarantee that the last row of the tower witnesses that M^* is a (μ, ω) -limit model. This will be done by imposing the following ordering on rows of the array:

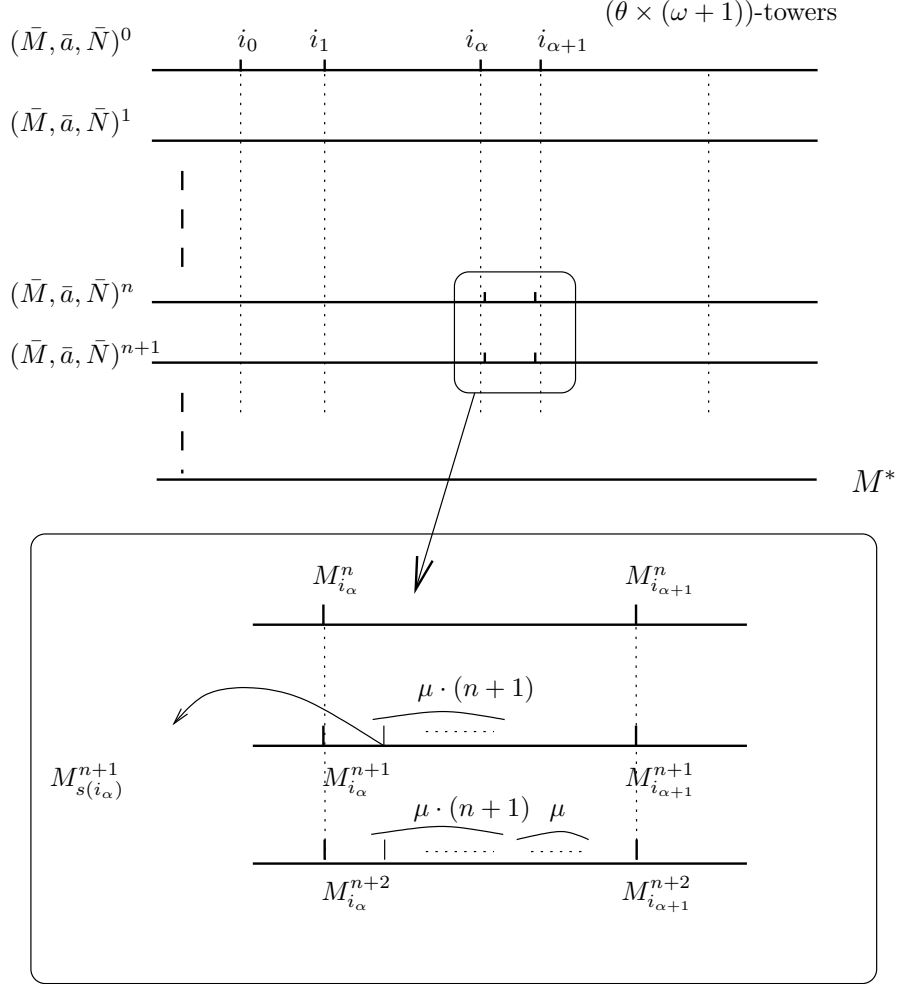
Definition 5.1. For towers $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}$ and $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}_{\mu, I'}$ with $I \subseteq I'$, we write $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$ if and only if for every $i \in I$, $a_i = a'_i$, $N_i = N'_i$ and M'_i is universal over M_i .

Remark 5.2. The ordering $<$ on towers is identical to the ordering $<^c_\mu$ defined in [ShVi]. The superscript was used by Shelah and Villaveces to distinguish this ordering from others. We only use one ordering on towers, so we omit the superscripts and subscripts here.

To get a relatively full tower at the end of the construction, we will require that at stage n of our construction the tower that we build is indexed by I_n described here:

Notation 5.3. Fix an increasing and continuous chain of well-ordered sets $\langle I_n \mid n \leq \omega \rangle$ and an increasing and continuous sequence of elements $\langle i_\alpha \mid \alpha \leq \theta \rangle$ such that each I_n has a supremum i_θ and $\langle i_\alpha \mid \alpha < \theta \rangle$ is cofinal in each $I_n \setminus \{i_\theta\}$. We additionally require that $\text{otp}(\{j \in I_n \mid i_\alpha < j < i_{\alpha+1}\})$ is $\mu \cdot n$ for each $\alpha < \theta$ and each $n \leq \omega$. An example of such $\langle I_n \mid n \leq \omega \rangle$ is $I_n = \theta \times (\mu \cdot n) \cup \{i_\theta\}$ ordered lexicographically, where i_θ is an element \geq each $i \in \bigcup_{n < \omega} I_n$.

Here we depict the construction:



We will build this array row-by-row by constructing towers by induction on $n < \omega$. We verify that it is possible to carry out the successor step of the construction. This is a particular case of Theorem II.7.1 of [Va]. But since our context is somewhat easier, we do not encounter so many obstacles as in [Va] and we provide a different, more direct proof here:

Theorem 5.4 (Dense $<$ -extension property). *Given $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}_{\mu, I_n}^*$ there exists $(\bar{M}, \bar{a}, \bar{N})^{n+1} \in \mathcal{K}_{\mu, I_{n+1}}^*$ such that $(\bar{M}, \bar{a}, \bar{N})^n < (\bar{M}, \bar{a}, \bar{N})^{n+1}$ and for each $(p, N) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j), N_j)$ and (p, N) are parallel.*

Before we prove Theorem 5.4, we prove a slightly weaker extension property:

Lemma 5.5 ($<$ -extension property). *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ for any $n < \mu^+$, there exists a $<$ -extension $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ of $(\bar{M}, \bar{a}, \bar{N})$ such that for each $i \in I$, $M'_{\text{succ}_I(i)}$ is a $(\mu, \mu \cdot n)$ -limit model over M'_i .*

Proof of Lemma 5.5. Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ we will define a $<$ extension $(\bar{M}', \bar{a}, \bar{N})$ by a directed system by induction on $i \in I$. We define M_i^+ and a directed system of $\prec_{\mathcal{K}}$ -embeddings $\langle f_{i,j} \mid i < j \in I \rangle$ such that for $i \in I$, $M_i \prec_{\mathcal{K}} M_i^+$ for $i \leq j$, $f_{i,j} : M_i^+ \rightarrow M_j^+$ and $f_{i,j} \upharpoonright M_i = \text{id}_{M_i}$. We further require that $M_{\text{succ}_I(i)}^+$ is a $(\mu, \mu \cdot n)$ -limit model over $f_{i, \text{succ}_I(i)}(M_i^+)$ and $\text{ga-tp}(a_i/f_{i, \text{succ}_I(i)}(M_i^+))$ does not μ -split over N_i .

This construction is done by induction on $i \in I$ using both the disjoint amalgamation property and the existence of non- μ -splitting extensions. At limit stages we take direct limits so that $f_{j,i} \upharpoonright M_j = \text{id}_{M_j}$. This is possible by Subclaims II.7.10 and II.7.11 of [Va] or see Claim 2.17 of [GrVa2]. If the direct limit M' is not universal over M_i , simply take an extension of both M' and M_i which is universal over M_i and call this M_i^+ . We forfeit continuity of the tower at this point, but it will be recovered later using reduced towers.

Let $f_{j, \text{sup}\{I\}}$ and $M_{\text{sup}\{I\}}$ be the direct limit of this system such that $f_{j, \text{sup}\{I\}} \upharpoonright M_j = \text{id}_{M_j}$. We can now define $M'_j := f_{j, \text{sup}\{I\}}(M_j^+)$ for each $j \in I$. The details of the verification that $(\bar{M}', \bar{a}, \bar{N})$ are as required are left to the reader, but can also be found in [Va]. \dashv

Proof of Theorem 5.4. Given $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}_{\mu, I_n}^*$, let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_n}^*$ be an extension of $(\bar{M}, \bar{a}, \bar{N})$ as in Lemma 5.5.

For each i_α , let $\langle M'_j \mid j \in I_{n+1}, i_\alpha < j < i_{\alpha+1} \rangle$ witness that $M'_{\text{succ}_{I_n}(i)}$ is a $(\mu, \mu \cdot n)$ -limit model over M'_i . Without loss of generality we may assume that each M'_j is a limit model over its predecessor.

Fix $\{(p, N)_{i_\alpha}^j \mid i_\alpha + \mu \cdot n < j < i_{\alpha+1}\}$ an enumeration of $\bigcup \{\text{St}(M_i) : i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$. By our choice of I_{n+1} and stability in μ , such an enumeration is possible. Since $M'_{\text{succ}_{I_{n+1}}(j)}$ is universal over M'_j , there exists a realization in $M'_{\text{succ}_{I_{n+1}}(j)}$ of the non- μ -splitting extension of $p_{i_\alpha}^j$ to M'_j .

Let a_j be this realization and take $N'_j := N_{i_\alpha}^j$.

Notice that $(\langle M'_j \mid j \in I_{n+1} \rangle, \langle a_j \mid j \in I_{n+1} \rangle, \langle N_j \mid j \in I_{n+1} \rangle)$ provide the desired extension of $(\bar{M}, \bar{a}, \bar{N})$ in $\mathcal{K}_{\mu, I_{n+1}}^*$. \dashv

We are almost ready to carry out the complete construction. However, notice that Theorem 5.4 does not provide us with a continuous extension. Therefore the bottom row of our array may not be continuous at θ which would prevent us from applying Fact 4.4 to conclude that M^* is a (μ, θ) -limit model. So we will further require that the towers that occur in our array are all continuous. This can be guaranteed by restricting ourselves to reduced towers as in [ShVi] and [Va].

Definition 5.6. A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is said to be *reduced* provided that for every $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ with $(\bar{M}, \bar{a}, \bar{N}) \leq_{\mu, I} (\bar{M}', \bar{a}, \bar{N})$ we have

that for every $i \in I$,

$$(*)_i \quad M'_i \cap \bigcup_{j \in I} M_j = M_i.$$

If we take a $<$ -increasing chain of reduced towers, the union will be reduced. The following fact appears as Theorem 3.1.14 of [ShVi]. We provide the proof for completeness.

Fact 5.7. *If $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}_{\mu, I_\gamma}^* \mid \gamma < \beta \rangle$ is $<$ -increasing and continuous sequence of reduced towers, then the union of this sequence of towers is a reduced tower.*

Proof. Denote by $(\bar{M}, \bar{a}, \bar{N})^\beta$ the union of the sequence of towers and $I_\beta := \bigcup_{\gamma < \beta} I_\gamma$. Then $\bar{a}^\beta = \bar{a}^0$, $\bar{N}^\beta = \bar{N}^0$ and $\bar{M}^\beta = \langle M_i^\beta \mid i \in \bigcup_{\gamma < \beta} I_\gamma \rangle$ where $M_i^\beta = \bigcup_{\gamma < \beta} M_i^\gamma$.

Suppose that $(\bar{M}, \bar{a}, \bar{N})^\beta$ is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_\beta}^*$ witness this. Then there exists an $i \in I_\beta$ and an element b such that $b \in (M'_i \cap \bigcup_{j \in I_\beta} M_j^\beta) \setminus M_i^\beta$. There exists $\gamma < \beta$ such that $b \in \bigcup_{j \in I_\gamma} M_j^\gamma \setminus M_i^\gamma$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \upharpoonright I_\gamma$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is not reduced. \dashv

The following appears in [ShVi] (Theorem 3.1.13).

Fact 5.8 (Density of reduced towers). *There exists a reduced $<$ -extension of every tower in $\mathcal{K}_{\mu, I}^*$.*

Proof. Suppose for the sake of contradiction that no $<$ -extension of $(\bar{M}, \bar{a}, \bar{N})$ is reduced. This allows us to construct a \leq -increasing and continuous sequence of towers $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \in \mathcal{K}_{\mu, I}^* \mid \zeta < \mu^+ \rangle$ such that $(\bar{M}, \bar{a}, \bar{N})^{\zeta+1}$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\zeta$ is not reduced.

The construction is done inductively in the obvious way:

For each $b \in \bigcup_{\zeta < \mu^+, i \in I} M_i^\zeta$ define

$$i(b) := \min \{ i \in I \mid b \in \bigcup_{\zeta < \mu^+} \bigcup_{j < i} M_j^\zeta \} \text{ and}$$

$$\zeta(b) := \min \{ \zeta < \mu^+ \mid b \in M_{i(b)}^\zeta \}.$$

$\zeta(\cdot)$ can be viewed as a function from μ^+ to μ^+ . Thus there exists a club $E = \{ \delta < \mu^+ \mid \forall b \in \bigcup_{i < \in I} M_i^\delta, \zeta(b) < \delta \}$. Actually, all we need is for E to be non-empty.

Fix $\delta \in E$. By construction $(\bar{M}, \bar{a}, \bar{N})^{\delta+1}$ witnesses the fact that $(\bar{M}, \bar{a}, \bar{N})^\delta$ is not reduced. So we may fix $i \in I$ and $b \in M_i^{\delta+1} \cap \bigcup_{j \in I} M_j^\delta$ such that $b \notin M_i^\delta$. Since $b \in M_i^{\delta+1}$, we have that $i(b) \leq i$. Since $\delta \in E$, we know that there exists $\zeta < \delta$ such that $b \in M_{i(b)}^\zeta$. Because $\zeta < \delta$ and $i(b) < i$, this implies that $b \in M_i^\delta$ as well. This contradicts our choice of i and b witnessing the failure of $(\bar{M}, \bar{a}, \bar{N})^\delta$ to be reduced. \dashv

Theorem 5.9 (Reduced towers are continuous). *If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is reduced, then it is continuous.*

By a slight revision to Lemma 5.5, we can conclude:

Lemma 5.10. *Suppose that $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is reduced, then for every $I_0 \subseteq I$, $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright I_0$ is reduced.*

Proof of Theorem 5.9. Suppose the claim fails for μ . Let δ be the minimal limit ordinal such that there exists an I of order type α and $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ a reduced tower discontinuous at δ . Without loss of generality $I = \alpha$. We are assuming that $M_\delta \not\prec_{\mathcal{K}} \bigcup_{i < \delta} M_i$.

We can apply Lemma 5.10, to assume that $\alpha = \delta + 1$. Fix $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \delta+1}^*$ reduced and discontinuous at δ . Let $b \in M_\delta \setminus \bigcup_{i < \delta} M_i$ be given. Let M^b be a model of cardinality μ inside \mathfrak{C} containing $\bigcup_{i < \delta} M_i^i \cup \{b\}$.

Claim 5.11. *There exists a $<$ -extension of $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$, say $(\bar{M}', \bar{a} \upharpoonright \delta, \bar{N} \upharpoonright \delta) \in \mathcal{K}_{\mu, \delta}^*$ containing b .*

Let $M'_\delta \prec_{\mathcal{K}} \mathfrak{C}$ be a limit model universal over M_δ containing $\bigcup_{i < \delta} M_i'$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \delta+1}^*$ is an extension of $(\bar{M}, \bar{a}, \bar{N})$ witnessing that $(\bar{M}, \bar{a}, \bar{N})$ is not reduced.

Proof of Claim 5.11. We use the minimality of δ and the density of reduced towers to build a $<$ -increasing and continuous sequence of reduced (and continuous) towers $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \mid \zeta < \delta \rangle$ such that $(\bar{M}, \bar{a}, \bar{N})^0 := (\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$. This gives us a δ by δ array of models. If b appears in this array, we are done. So let us suppose that $\text{ga-tp}(b / \bigcup_{i < \delta} M_i^i)$ is non-algebraic. Since $\bigcup_{i < \delta} M_i^i$ is a (μ, δ) -limit model (witnessed by the diagonal of this array), there exists $\xi < \delta$ such that $\text{ga-tp}(b / \bigcup_{i < \delta} M_i^i)$ does not μ -split over M_ξ^ξ .

We will find a $<$ -extension of $(\bar{M}, \bar{a}, \bar{N})$ by defining an $\prec_{\mathcal{K}}$ -increasing chain of models $\langle N_i^* \mid i < \alpha \rangle$ and an increasing chain of $\prec_{\mathcal{K}}$ -mappings $\langle h_i \mid i < \delta \rangle$ with the intention that the pre-image of N_i^* under an extension of $\bigcup_{i < \delta} h_i$ will form a sequence \bar{M}^* such that $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}^*, \bar{a}, \bar{N})$, $b \in M_{\xi+1}^*$ and $M_i^* = M_i^i$ for all $i < \xi$. We choose by induction on $i < \delta$ a $\prec_{\mathcal{K}}$ -increasing and continuous chain of limit models $\langle N_i^* \in \mathcal{K}_\mu \mid i < \delta \rangle$ and an increasing and continuous sequence of $\prec_{\mathcal{K}}$ -mappings $\langle h_i \mid i < \delta \rangle$ satisfying

- (1) N_{i+1}^* is a limit model and is universal over N_i^*
- (2) $h_i : M_i^i \rightarrow N_i^*$
- (3) $h_i(M_i^i) \prec_{\mathcal{K}} M_i^{i+1}$
- (4) $\text{ga-tp}(h_{i+1}(a_i) / N_i^*)$ does not μ -split over $h_i(N_i)$
- (5) $M^b \prec_{\mathcal{K}} N_{\xi+1}^*$ and
- (6) for $i \leq \xi$, $N_i^* = M_i^i$ with $h_i = \text{id}_{M_i^i}$.

The requirements determine the definition of N_i^* for $i \leq \xi$. We proceed with the rest of the construction by induction on $i < \delta$. If i is a limit ordinal $\geq \xi$, let $N_i^* = \bigcup_{j < i} N_j^*$ and $h_i = \bigcup_{j < i} h_j$.

Suppose that we have defined h_i and N_i^* satisfying the conditions of the construction. We now describe how to define N_{i+1}^* . First, we extend h_i to $\bar{h}_i \in \text{Aut}(\mathfrak{C})$. We can assume that $\bar{h}_i(a_i) \in M_{i+1}^{i+2}$. This is possible since M_{i+1}^{i+2} is universal over $h_i(M_i^i)$ by construction (as M_{i+1}^{i+2} is universal over $M_i^{i+2} \succ_{\mathcal{K}} h_i(M_i^i)$).

Since $\text{ga-tp}(a_i/M_i^i)$ does not μ -split over N_i , we have that $\text{ga-tp}(\bar{h}_i(a_i)/h_i(M_i^i))$ does not μ -split over $h_i(N_i)$. We now adjust the proof of the existence property for non-splitting extensions.

Claim 5.12. *We can find $g \in \text{Aut}(\mathfrak{C})$ such that $\text{ga-tp}(g(\bar{h}_i(a_i))/N_i^*)$ does not μ -split over $h_i(N_i)$, $g(\bar{h}_i(M_{i+1}^{i+1})) \prec_{\mathcal{K}} M_{i+1}^{i+2}$ and such that $g \upharpoonright (h_i(M_i^i)) = \text{id}_{h_i(M_i^i)}$.*

Proof of Claim 5.12. First we find a $\prec_{\mathcal{K}}$ -mapping f such that $f : N_i^* \rightarrow h_i(M_i^i)$ such that $f \upharpoonright h_i(N_i) = \text{id}_{h_i(N_i)}$ which is possible since $h_i(M_i^i)$ is universal over $h_i(N_i)$. Notice that $\text{ga-tp}(f^{-1}(\bar{h}_i(a_i))/N_i^*)$ does not μ -split over $h_i(N_i)$ and

$$(+) \quad \text{ga-tp}(f^{-1}(\bar{h}_i(a_i))/h_i(M_i^i)) = \text{ga-tp}(\bar{h}_i(a_i)/h_i(M_i^i))$$

by a non-splitting argument as in the proof of Fact 2.3.

Let N^+ be a limit model of cardinality μ containing $f^{-1}(\bar{h}_i(a_i))$ with $f^{-1}(\bar{h}_i(M_{i+1}^{i+1})) \prec_{\mathcal{K}} N^+$. Now using the equality of types (+) and the fact that M_{i+1}^{i+2} is universal over $h_i(M_i^i)$ with $\bar{h}_i(a_i) \in M_{i+1}^{i+2}$, we can find a $\prec_{\mathcal{K}}$ -mapping $f^+ : N^+ \rightarrow M_{i+1}^{i+2}$ such that $f^+ \upharpoonright h_i(M_i^i) = \text{id}_{h_i(M_i^i)}$ and $f^+(f^{-1}(\bar{h}_i(a_i))) = \bar{h}_i(a_i)$. Now set $g := f^+ \circ f^{-1} : \bar{h}_i(M_{i+1}^{i+1}) \rightarrow M_{i+1}^{i+2}$. \dashv

Fix such a g as in the claim and set $h_{i+1} := g \circ \bar{h}_i \upharpoonright M_{i+1}^{i+1}$. Let N_{i+1}^* be a $\prec_{\mathcal{K}}$ extension of N_i^* , M^b and $h_{i+1}(M_{i+1}^{i+1})$ of cardinality μ inside \mathfrak{C} . Furthermore, choose N_{i+1}^* to be a limit model and universal over N_i^* .

This completes the construction.

We now argue that the construction of these sequences is enough to find a \prec -extension, $(\bar{M}^*, \bar{a}, \bar{N})$, of $(\bar{M}, \bar{a}, \bar{N})$ such that $b \in M_\zeta^*$ for some $\zeta < \delta$.

Let $h_\delta := \bigcup_{i < \delta} h_i$. We will be defining for $i < \delta$, M_i^* to be pre-image of N_i^* under some extension of h_δ . The following claim allows us to choose the pre-image so that M_ζ^* contains b for some $\zeta < \delta$.

Claim 5.13. *There exists $h \in \text{Aut}(\mathfrak{C})$ extending $\bigcup_{i < \delta} h_i$ such that $h(b) = b$.*

Proof of Claim 5.13. Let $h_\delta := \bigcup_{i < \delta} h_i$. Consider the increasing and continuous sequence $\langle h_\delta(M_i^i) \mid i < \delta \rangle$. (This sequence is increasing by our choice of g in 5.12: if $d \in M_i^i$, then $h_{i+1}(d) = g(\bar{h}_i(d)) = g(h_i(d)) = h_i(d)$, by the definitions of h_{i+1} , \bar{h}_i and g .) By invariance, $h_\delta(M_{i+1}^{i+1})$ is universal over $h_\delta(M_i^i)$ and each $h_\delta(M_i^i)$ is a limit model.

Furthermore, from our choice of ξ , we know that $\text{ga-tp}(b/M_i^\delta)$ does not μ -split over M_ξ^δ . Since $h_i(M_i^i) \prec_{\mathcal{K}} M_i^{i+1} \prec_{\mathcal{K}} \bigcup_{j < \delta} M_j^\delta$, monotonicity of

non-splitting allows us to conclude that

$$\text{ga-tp}(b/h_\delta(M_i^i)) \text{ does not } \mu\text{-split over } M_\xi^\xi.$$

This allows us to apply Assumption 2.4, to $\text{ga-tp}(b/\bigcup_{i<\delta} h_\delta(M_i^i))$ yielding

$$(**) \quad \text{ga-tp}(b/\bigcup_{i<\delta} h_\delta(M_i^i)) \text{ does not } \mu\text{-split over } M_\xi^\xi.$$

We can extend $\bigcup_{i<\delta} h_i$ to an automorphism h^* of \mathfrak{C} . We will first show that

$$(***) \quad \text{ga-tp}(b/h^*(\bigcup_{i<\delta} M_i^i)) = \text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M_i^i)).$$

By invariance and our choice of ξ in (*),

$$\text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M_i^i)) \text{ does not } \mu\text{-split over } M_\xi^\xi.$$

We will use non-splitting to derive (***). In accordance with the definition of splitting, let $N^1 = \bigcup_{i<\delta} M_i^i$, $N^2 = h^*(\bigcup_{i<\delta} M_i^i)$ and $p = \text{ga-tp}(b/N^2)$. By (**), we have that $p \upharpoonright N^2 = h^*(p \upharpoonright N^1)$. In other words,

$$\text{ga-tp}(b/h^*(\bigcup_{i<\delta} M_i^i)) = \text{ga-tp}(h^*(b)/h^*(\bigcup_{i<\delta} M_i^i)),$$

as desired.

From (***), we can find an automorphism f of \mathfrak{C} such that $f(h^*(b)) = b$ and $f \upharpoonright h^*(\bigcup_{i<\delta} M_i^i) = \text{id}_{h^*(\bigcup_{i<\delta} M_i^i)}$. Notice that $h := f \circ h^*$ satisfies the conditions of the claim. +

Now that we have a automorphism h fixing b and $\bigcup_{i<\delta} M_i$, we can define for each $i < \delta$, $M_i^* := h^{-1}(N_i^*)$.

Claim 5.14. $(\bar{M}^*, \bar{a}, \bar{N})$ is a $<$ -extension of $(\bar{M}, \bar{a}, \bar{N})$ such that $b \in M_{\xi+1}^*$.

Proof of Claim 5.14. By construction $b \in M_\delta^\delta \subseteq N_{\xi+1}^*$. Since $h(b) = b$, this implies $b \in M_{\xi+1}^*$. To verify that we have a \leq -extension we need to show for $i < \delta$:

- i. M_i^* is universal over M_i
- ii. $a_i \in M_{i+1}^* \setminus M_i$ for $i+1 < \delta$ and
- iii. $\text{ga-tp}(a_i/M_i^*)$ does not μ -split over N_i whenever $i, i+1 \leq \delta$.

Item i. follows from the fact that M_i^i is universal over M_i and $M_i^i \prec_{\mathcal{K}} M_i^*$. Item iii. follows from invariance and our construction of the N_i^* 's. Finally, recalling that a non-splitting extension of a non-algebraic type is also non-algebraic, we see that Item iii implies $a_i \notin M_i^*$. By our choice of $h_{i+1}(a_i) \in M_{i+1}^{i+2} \prec_{\mathcal{K}} N_{i+1}^*$, we have that $a_i \in M_{i+1}^*$. Thus Item ii is satisfied as well. +

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⊣

Corollary 5.15. *In Theorem 5.4, we can choose $(\bar{M}, \bar{a}, \bar{N})^{\alpha+1}$ to be continuous.*

The construction:

Define by induction on $n \leq \omega$ a $<$ -increasing and continuous chain of reduced towers $(M, \bar{a}, \bar{N})^n \in \mathcal{K}_{\mu, I_n}^*$ such that $(M, \bar{a}, \bar{N})^n$ is “dense in” $(\bar{M}, \bar{a}, \bar{N})^{n+1}$ in the sense of Theorem 5.4.

Corollary 5.15 tells us that the construction is possible in successor cases. In the limit case, let $I_\omega = \bigcup_{m < \omega} I_m$, and simply define, for $i \in I_\omega$,

$$M_i^\omega = \bigcup_{n \geq m} M_i^n,$$

where (say) $i \in I_m$. To see that the construction suffices for what we need, first notice that the last column of the array, $\langle M_{i_\alpha}^n \mid n < \omega \rangle$, witnesses that M^* is a (μ, ω) -limit model. In light of Fact 4.4 we need only verify that the last row of the array is a relatively full tower of cofinality θ .

Claim 5.16. *$(\bar{M}, \bar{a}, \bar{N})^\omega$ is full relative to $\langle M_i^n \rangle_{n < \omega, i \in I_\omega}$.*

Proof. Given i with $i_\alpha \leq i < i_{\alpha+1}$, let (p, M_i^n) be some strong type in $\mathfrak{St}(M_i^\omega)$. Notice that $(p \upharpoonright M_i^{n+1}, M_i^n) \in \mathfrak{St}(M_i^{n+1})$. By construction there is a $j \in I_\omega^{n+1}$ with $i \leq j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_{i+j}/M_{i+j}^{n+2}), N_{i+j}^{n+2})$ is parallel to $p \upharpoonright M_i^{n+1}$. We will show that $(\text{ga-tp}(a_{i+j}/M_{i+j}^\omega), N_{i+j}^\omega)$ is parallel to (p, N) .

First notice that $\text{ga-tp}(a_{i+j}/M_{i+j}^\omega)$ does not μ -split over $N_{i+j}^\omega = N_{i+j}^{n+2}$. Since $(\text{ga-tp}(a_{i+j}/M_{i+j}^{n+2}), N_{i+j}^{n+2})$ is parallel to $(p \upharpoonright M_i^{n+1}, M_i^n)$ there is $q \in \text{ga-S}(M_{i+j}^\omega)$ such that q extends both $p \upharpoonright M_i^{n+1}$ and $\text{ga-tp}(a_{i+j}/M_{i+j}^{n+2})$. By two separate applications of the uniqueness of non- μ -splitting extensions we know that $q \upharpoonright M_i^\omega = p$ and $q = \text{ga-tp}(a_{i+j}/M_{i+j}^\omega)$. To see that (q, N_{i+j}^ω) is parallel to (p, M_i^n) , let M' be an extension of M_{i+j}^ω of cardinality μ . Since $(p \upharpoonright M_i^{n+1}, M_i^n)$ and $(q \upharpoonright M_{i+j}^{n+2}, N_{i+j}^{n+2})$ are parallel, there is $q' \in \text{ga-S}(M')$ extending both $p \upharpoonright M_i^{n+1}$ and $q \upharpoonright M_{i+j}^{n+2}$ and not μ -splitting over both M_i^n and N_{i+j}^{n+2} . By the uniqueness of non- μ -splitting extensions, we have that q' is also an extension of q and p . Thus q witnesses that (q, N_{i+j}^ω) and (p, M_i^n) are parallel. ⊣

This completes the proof of Theorem 1.15.

Remark 5.17. The same proof provides uniqueness of (μ, σ) -limit models for $\sigma \geq \kappa_\mu(\mathcal{K})$ where $\kappa_\mu(\mathcal{K})$ is the minimal ordinal κ such that if $\langle M_i \mid i < \alpha \rangle$ is a $<_{\mathcal{K}}$ -increasing and continuous chain of limit models of cardinality μ and $\alpha \geq \kappa$, then for every $p \in \text{ga-S}(\bigcup_{i < \alpha})$ there exists $i < \alpha$ such that p does not μ -split over M_i . For tame μ -stable AECs, $\kappa_\mu(\mathcal{K})$ is bounded [GrVa1].

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