

Categoricity may fail late

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Abstract

We build an example that generalizes [HS90] to uncountable cases. In particular, our example yields a sentence $\psi \in \mathcal{L}_{(2^\lambda)^+, \omega}$ that is categorical in $\lambda, \lambda^+, \dots, \lambda^{+k}$ but not in $\beth_{k+1}(\lambda)^+$. This is connected with the Łoś Conjecture and with Shelah's own conjecture and construction of excellent classes for the $\psi \in \mathcal{L}_{\omega_1, \omega}$ case.

1 The Łoś Conjecture, without excellence

Early results on the Categoricity Spectrum launched the development of Stability Theory and Classification Theory for first order logic. In the natural quest for generalizing the powerful results of those theories to Nonelementary Classes, some questions on Categoricity – specifically, the status of the Łoś Conjecture (=Morley Theorem in First Order) – became a crucial test question. Among the specific issues studied, the following is central.

Question 1.1 Old folklore question: *what is the status of the Łoś Conjecture (= Morley Theorem in the first order case) for $\psi \in \mathcal{L}_{\omega_1, \omega}$?*

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Keisler in [Kei69] first provided a condition under which the Morley analysis works (and thus provides a positive answer to the question)). However, Marcus in [Mar72] showed that this condition is unfortunately not necessary.

In the landmark papers [She83a] and [She83b], Shelah provided a positive answer to this question under weak set theoretical assumptions such as weak versions of the GCH (specifically, weak diamonds). He identified the class of ‘excellent sentences’ and showed that if an $\mathcal{L}_{\omega_1\omega}$ -sentence ψ is excellent then the Łoś Conjecture holds for $\text{Mod}(\psi)$ (and the Morley analysis can be carried to this situation). In those two Shelah papers, the need for these assumptions seemed to be implied. Harrington explicitly voiced a natural question: he asked whether those assumptions were really needed (or in what sense could one weaken them). An answer to Harrington’s question appeared in [HS90]: they provide the first examples of $\mathcal{L}_{\omega_1\omega}$ sentences which were categorical up to some level but not excellent.

Excellent classes turned out to be of greater mathematical relevance than one would perhaps think. Not only did their definition provide the early conceptual tools for generalizing the work on categoricity to various nonelementary contexts, but also a version of the Main Gap was proved for them by Grossberg and Hart (see [GH89]), they are actually at the heart of Zil’ber’s recent study of fields with pseudoexponentiation and the Schanuel Conjecture (see [Zil]). An excellent expository paper that connects excellent classes to other domains in the classification theory for nonelementary classes is the ‘Bilgi’ paper by Grossberg ([Gro02]).

The ‘old folklore question’ of course admits many generalizations of the kind

Question 1.2 *What is (the status of) the Łoś Conjecture for $\psi \in L_{\kappa,\omega}$ (suitable κ)?*

What is (the status of) the Łoś Conjecture for \mathfrak{K} (say) a PC-class?

What is (the status of) the Łoś Conjecture for \mathfrak{K} a non-elementary class?

Of course, those questions (especially the last one) are way too general to be settled in short. In particular, the long series of papers [She87], [MS90], [KS96], [Shed], [Sheb], [Shec], [Shea], [Shef], [Shee] may be regarded as a (long and until now far from complete) organized attempt to solve the third of these generalizations. Or at least, an attempt to provide enough set theoretical or model theoretical hypotheses to obtain *positive* solutions to the Łoś Conjecture. These include various large cardinal hypotheses, weak diamond style hypotheses, on the set theoretical side, and amalgamation properties, existence of ‘large’ models, axioms for building ‘nonforking’ frames, etc. on the model theoretical side.

Clearly, the long-run intention is to obtain not just very general settings in which the Loś Conjecture (or some decent enough version of it) holds, but also to obtain more ‘dividing lines’ to pursue the program of classification theory for nonelementary classes. In this respect, the attempts done in the papers mentioned in the last paragraph all address what could be called ‘the first test question’.

On what one could call ‘the *negative* complementary perspective¹,’ there are so far very few published results. The paper by Hart and Shelah is one the most important of these. Section 6 of [She87] also addresses some of these ‘negative’ aspects. The purpose of looking at some of these negative cases must be seen in contrast with the positive results: [HS90] must be read with a [She83a] and [She83b] background.

Ideally, this paper should be read with a [She87] and [Sheb] background. As more positive results appear, it is natural to attempt to ‘fill the gap’ between positive and negative cases, and to provide new tools for understanding the blurry boundary between model (or set) theoretic situations where the Loś Conjecture holds or fails.

In this paper, we focus our energy in generalizations of Hart and Shelah’s counterexample to the Loś Conjecture in [HS90]. A second aim of this paper is to try to **explain** what can be expected (and why) on this ‘boundary region’ between positive and negative answers to the Loś Conjecture. The following table summarizes the previous remarks and the status of research on the subject. LC stands for the Loś Conjecture.

	FOL	$\mathcal{L}_{\omega_1\omega}$	$\mathcal{L}_{\lambda+\omega}$	AEC
Full LC holds	Morley	Shelah [She83a] and [She83b]	Shelah [Shee] and consequences	many partial results
Some LC fails	—	[HS90]	here	?

Finally, we thank Rami Grossberg, Alexei Kolesnikov and John Baldwin for several remarks concerning (directly or less directly) this work.

2 Toward the counterexample

We construct here a generalization of the proof by Hart and Shelah in [HS90], shifting the focus of their example from $\aleph_0, \aleph_1, \dots, \aleph_k$ to $\lambda, \lambda^+, \dots, \lambda^{+k}$ and on the way switching the logic for which the class of models is elementary from $\mathcal{L}_{\omega_1\omega}$ to $\mathcal{L}_{(2^\lambda)^+\omega}$.

¹Model theoretic phenomena dealing with cases where the Loś Conjecture fails

The construction first follows a path parallel to their older proof, and then takes care of new complications arising at the time of setting up the $\mathcal{P}^-(n)$ -diagrams (as in [She83a]) amalgamation family. The main differences with the construction in [HS90] are the need here for the use of a regular filter \mathfrak{D} and the replacement of ω there by much larger groups here.

As in [HS90], we first describe **canonical models** M_I , $M_{I,f}$ and in a second stage we extract from them the sentence ψ in $\mathcal{L}_{(2^\lambda)^+, \omega}$.

2.1 The construction

Context 2.1 *For this section we fix λ an infinite cardinal and $k < \omega$ ($k \geq 2$).*

The canonical models will be built using groups defined on the set S of finite subsets of λ .

Definition 2.2

1. Let $S = S_\lambda = [\lambda]^{<\aleph_0}$,
2. $\mathfrak{D} = \mathfrak{D}_\lambda = \{A \subset S \mid \exists u_A \in S \forall v \in S (u_A \subset v \rightarrow v \in A)\}$, the regular filter on S generated by sets of the form $\langle u \rangle = \{v \in S \mid u \subset v\}$.
3. $G^+ = G_\lambda^+ = {}^S(\mathbb{Z}_2)$, as a group with the natural operation $(f+g)(v) = f(v) +_{\mathbb{Z}_2} g(v)$,
4. $G = G_\lambda = \{f \in {}^S(\mathbb{Z}_2) \mid \ker(f) \in \mathfrak{D}\}$, as a subgroup of G^+ : $G \triangleleft G^+$, as if $f, g \in G$ then $\ker(f), \ker(g) \in \mathfrak{D}$, so $\ker(f+g) \supset \ker(f) \cap \ker(g) \in \mathfrak{D}$, so $\ker(f+g) \in \mathfrak{D}$ and $f+g \in G$. Note that $|G| = 2^\lambda$.

It is worth keeping in mind that the vocabulary for the construction of M_I and the idea of the definition of ψ depends on these basic notions.

Definition 2.3 The construction of the model M_I .

For a fixed set I we first define the following objects.

- (a) $H = H_I$ is the group with set of elements $[{}^k I]^{<\aleph_0}$, with ‘addition’ given by $S + T := S \Delta T$ (symmetric difference). (We will also regard sometimes $h \in H$ as a function ${}^k I \rightarrow \mathbb{Z}_2$, with the usual translation by means of the characteristic function of h ; clearly, this is in our case a function with value 0 except at finitely many places.)

(b) the model $M = M_I$:

Universe:

$$|M_I| = I \cup {}^k I \cup {}^{k+1} I \cup ({}^k I \times S \times H) \cup ({}^k I \times S \times \mathbb{Z}_2) \cup H \cup ({}^{k+1} I \times G).$$

One way of thinking about the model is as

$$|M_I|$$

consisting of

$$\overbrace{I \cup {}^k I \cup {}^{k+1} I}^{\text{'control part'}} \overbrace{\cup ({}^k I \times S \times H) \cup ({}^k I \times S \times \mathbb{Z}_2) \cup H \cup ({}^{k+1} I \times G)}^{\text{'zeroless copies of } H_I, \mathbb{Z}_2 \text{ and } G}.$$

Notice that we actually get an empty intersection between all those pieces of the model.

Before putting structure on $|M_I|$, we try to provide the reader with a general description of how our model will be. The relations and functions below will make specific the following vague description: in addition to the indices I and k and $k + 1$ tuples of indices from the ‘control region’ and a copy of H , we have ‘zeroless versions’ of the groups H_I , \mathbb{Z}_2 and a copy of G for each $k + 1$ -tuple, one for each k -tuple from I and finite set $s \in S$.

In a way analog to [HS90], we will get the interesting behavior in the models by predicates Q_s connecting (for each $s \in S$) one copy of H with k copies of \mathbb{Z}_2 and one copy of G .

Remark: these functions capture our idea of building the model with ‘shifted’ copies of the groups G , H_I and \mathbb{Z}_2 – through maps between these groups and ${}^k I$.

Basic Relations: $P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M, (P_{2,s}^M)_{s \in S}, P_3^M, (P_{3,s}^M)_{s \in S}, P_4^M, P_5^M$.

These are defined by

$$P_0^M = I,$$

$$P_{1,1}^M = {}^k I,$$

$$P_{1,2}^M = {}^{k+1} I,$$

$$P_2^M = {}^k I \times S \times H \text{ (a copy of } H \text{ for each } b \in {}^k I, s \in S \text{ - recall that } H = [{}^k I]^{\aleph_0}),$$

$$\text{for } s \in S, P_{2,s}^M = \{(u, s, h) \in P_2^M \mid u \in {}^k I, h \in H\} = {}^k I \times \{s\} \times H,$$

$$\begin{aligned}
 P_3^M &= {}^k I \times S \times \mathbb{Z}_2 \text{ (a copy of } \mathbb{Z}_2 \text{ for each } b \in {}^k I, s \in S), \\
 \text{for } s \in S, P_{3,s}^M &= \{(u, s, i) \in P_3^M \mid u \in {}^k I, i \in \mathbb{Z}_2\} = {}^k I \times \{s\} \times \mathbb{Z}_2, \\
 P_4^M &= H, \\
 P_5^M &= {}^{k+1} I \times G
 \end{aligned}$$

The intended meanings are:

$$\begin{aligned}
 P_2^M &\approx (H_{v,s})_{v \in {}^k I, s \in S}, \\
 P_3^M &\approx ((\mathbb{Z}_2)_{v,s})_{v \in {}^k I, s \in S}, \\
 P_5^M &\approx (G_u)_{u \in {}^{k+1} I}.
 \end{aligned}$$

Partial Functions: $(\pi_\ell)_{\ell < k}, F_2^M, F_3^M, F_4^M, F_5^M, (F_{3,g^*}^M)_{g^* \in G}$, given by

For $\ell < k$, the projections $\pi_\ell : {}^k I \rightarrow I$ given by

$$\pi_\ell(\bar{a}) = a_\ell$$

and $\pi_k : {}^{k+1} I \rightarrow I$ given by

$$\pi_k(\bar{a}) = a_k.$$

A unary function F_2^M with domain P_2^M , given by

$$F_2^M(u, s, h) = u,$$

A unary function F_3^M with domain P_3^M , given by

$$F_3^M(u, s, i) = u,$$

A unary function F_5^M with domain P_5^M , given by

$$F_5^M(u, g) = u,$$

A binary function F_4^M with domain $P_2^M \times P_4^M$, given (on our intended ‘zerless copies of H ’) by

$$F_4^M((v, s, h), h_1) = (v, s, h +_H h_1)$$

for $g^* \in G$, a unary function F_{3,g^*}^M with domain P_5^M , given by

$$F_{3,g^*}^M(u, g) = (u, g^* + g),$$

The intended meanings are a mixture of projections and functions that tell us ‘in which’ copy we are. Also notice that

$F_4^M(F_4^M((u, s, h), h_1), h_2) = F_4^M((u, s, h), h_3)$ if and only if $H \models h_1 + h_2 = h_3$. Hence, $+^H$ is definable.

Notice also that if we replace \mathbb{Z}_2 by a larger group, we may need an analog to F_4 for copies of \mathbb{Z}_2 .

A predicate Q_s , for each $s \in S$, that we shall relate later to $\mathcal{P}^-(n)$ -diagrams and their amalgamation, speaking about the ‘sth copy of one copy of H , k copies of \mathbb{Z}_2 and one copy of G ’. It is interpreted in M_I as the set of tuples

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_0, \dots, x_{k-1}, y_k, z \rangle$$

satisfying (for fixed $s \in S$!) $h_k \in H$, $i_\ell \in \mathbb{Z}_2 (\ell \leq k)$, $g \in G$:

(α) $a_\ell \in I$ with no repetitions ($\ell \leq k$),

(β) $u_\ell = \langle a_m | m \neq \ell \rangle \in P_{1,1}^M (\ell \leq k)$,

(γ) $y_k = (u_k, s, h_k) \in P_2^M$,

(δ) x_ℓ has the form $(u_\ell, s, i_\ell) \in P_3^M (\ell < k)$ so $i_\ell \in \mathbb{Z}_2$,

(ϵ) z is of the form $(u, g) \in P_5^M$, where $u = (a_0, \dots, a_k) \in {}^{k+1}I$ and

(ζ) (**main point**)

$$\mathbb{Z}_2 \models \sum_{\ell < k} i_\ell = h_k(u_0) + g(s).$$

Part (ζ) of the definition provides the connection between k copies of \mathbb{Z}_2 , one copy of H , one copy of G and the $k+1$ k -element subsets of a set of size $k+1$ in I .

The fact that in (ζ) we choose 0 as subindex for u is not important; it could have been any $\ell \leq k$; note that u_k appears only in y_k .

We want to show that there are few M quite equivalent to M_I in the relevant sizes, and we shall give a full characterization of these models.

Now, we direct our attention to the language, in order to extract our sentence ψ .

Definition 2.4 Let τ^- be the vocabulary implicit in all the construction above, without including $\{Q_s | s \in S\}$ and τ be the full vocabulary implicit above. Specifically,

$$\begin{aligned} \tau^- &= \langle P_0, P_{1,1}, P_{1,2}, P_2, (P_{2,s})_{s \in S}, P_3, (P_{3,s})_{s \in S}, P_4, P_5, \\ &\quad \pi_0, \dots, \pi_{k-1}, F_2, F_3, F_4, F_5, (F_{3,g^*})_{g^* \in G} \rangle \\ \tau &= \tau^- \cup \{Q_s | s \in S\}. \end{aligned}$$

Notice that $|\tau| = |G| + |H| + |S| + \aleph_0 = 2^\lambda + |I|$, since $|G_\lambda| = 2^\lambda$.

Definition 2.5

For a set I and a function

$$f : {}^{k+1}I \times S \rightarrow \mathbb{Z}_2$$

we define a model $M_{I,f}$ as follows:

Vocabulary: also τ . The model will then be like M_I , only the interpretation of Q_s changes. So, we let

$$M_{I,f} \upharpoonright \tau^- = M_I \upharpoonright \tau^-,$$

and in $M_{I,f}$ the interpretation of $Q_s^{M_{I,f}}$ is the set of tuples

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_0, \dots, x_{k-1}, y_k, z \rangle$$

as in 2.3, only equation (ζ) becomes here

$$(\zeta)_f^* \quad \mathbb{Z}_2 \models \sum_{\ell < k} i_\ell = h_k(u_0) + g(s) + f(u, s).$$

$f(u, s)$ then gives us the ‘correction’ for any other copy of the groups.

$M_{I,f}^-$ is defined like M_I but omitting the predicates Q_s . Its vocabulary is τ^- from 2.4.

M is **strongly standard** if $M \upharpoonright \tau^- = M_I \upharpoonright \tau^-$ for $I = P_0^M$,

We leave the following claim with no proof.

Claim 2.6 *For any $M \models \psi$, $M_I \approx M_{I,0}$.*

Claim 2.7 *Every $M \models \psi$ is isomorphic to a strongly standard M .*

Next, a straightforward observation.

Claim 2.8 *$M_{I,f}$ is strongly standard.*

Definition 2.9 (The first class and the sentence)

1. $K_1 := \{M \mid M \approx M_{I,f} \text{ for some infinite set } I, \text{ for some } f \text{ as in 2.5}\}$ so K_1 is a class of τ -models, the vocabulary τ from 2.5.
2. Our sentence $\psi \in \mathcal{L}_{(2^\lambda)^+, \omega}(\tau)$ is defined by using the following sentences:
 - (a) T_0 consists of **all** the first order sentences which every M_I satisfies (as I is infinite, all the M_I are elementarily equivalent in first order),

- (b) $\psi_G \equiv \forall z_1 z_2 ([P_5(z_1) \wedge P_5(z_2) \wedge F_5(z_1) = F_5(z_2)] \rightarrow \bigvee_{g^* \in G} F_{3,g^*}(z_1) = z_2)$,
- (c) $\psi_{\mathbb{Z}_2} \equiv \forall y (P_2(y) \leftrightarrow \bigvee_{s \in S} P_{2,s}(y))$,
- (d) $\psi_H \equiv \forall y (P_3(y) \leftrightarrow \bigvee_{s \in S} P_{3,s}(y))$,

Then we define our sentence $\psi \in \mathcal{L}_{(2^\lambda)^+, \omega}(\tau)$ by

$$\psi \equiv \bigwedge T_0 \wedge \psi_G \wedge \psi_{\mathbb{Z}_2} \wedge \psi_H$$

ψ_G says that the copies of G are really copies of G , so that The construction of the models if used to prove that our sentence ψ has the desired properties. Note that, although there are 2^{2^λ} sentences in the logic, we are only using 2^λ of them, as witnessed by $|G| = 2^\lambda$.

- 3. $K_2 := \text{Mod}(\psi)$.
- 4. M from K_2 is **standard** if $P_{1,1}^M = [P_0^M]^k$ and $P_{1,2}^M = [P_0^M]^{k+1}$ and the π_ℓ 's are natural.

3 Recovering lost zeros

We now start the second stage of our proof, preparing the desired categoricity cases for ψ . For this, it is enough to show that every model in the suitable cardinals is isomorphic to a standard one. We will describe choices and correction functions, that will be used in counting the models in sizes $\lambda, \lambda^+, \dots, \lambda^{+m}$, $m < k$.

This comes down to trying to recover the ‘lost’ zero of the copies of the groups. To this end, we define ‘choices’ (depending on the model M and on various subsets of $P_{1,1}^M$) of relevant elements for the crucial equation, and ‘correction functions’ for these equations.

Being isomorphic to standard means that we can ‘make choices’ with zero corrections but we have freedom in choosing the choice function. Expanding choices from partial to global ones is the crux of the proof.

Definition 3.1

- 1. For $M \models \psi$, we say $(\bar{x}, \bar{y}, \bar{z})$ is a **partial M - (J_0, J_1, J_2) -choice** if
 - (a) $J_0, J_1 \subset P_{1,1}^M, J_2 \subset P_{1,2}^M$,
(the intended meaning here is for standard models: $J_0, J_1 \subset {}^k I, J_2 \subset {}^{k+1} I$)

(b) $\bar{x} = \langle x_{u,s} | s \in S, u \in J_0 \rangle$, where

$$x_{u,s} \in (\mathbb{Z}_2)_{u,s}^M := \{z \in P_{3,s}^M | F_3^M(u, s, z) = u\} \subset P_{3,s}^M.$$

(c) $\bar{y} = \langle y_{u,s} | s \in S, u \in J_1 \rangle$,

$$y_{u,s} \in H_{u,s}^M := \{y \in P_{2,s}^M | F_2^M(u, s, y) = u\} \subset P_{2,s}^M.$$

(d) $\bar{z} = \langle z_u | u \in J_2 \rangle$,

$$z_u \in G_u^M := \{z | F_5(z) = u\} \subset P_5^M.$$

(So, informally, \bar{x} chooses an element i in each copy of \mathbb{Z}_2 , \bar{y} chooses a h in each copy of H , \bar{z} chooses a g in each copy of G , for each relevant (u, s) , so $x_{u,s}$ is *some* element in the ‘fiber’ of u via F_3^M , and analogously for \bar{y} and \bar{z})

2. Call $(\bar{x}, \bar{y}, \bar{z})$ a **partial M - J -choice** if it is an M - (J, J, J_*^M) -choice, where

$$J_*^M := \left\{ a \in P_{1,2}^M \mid \bigwedge_{m \leq k} \exists b \in J \left[\bigwedge_{\ell < m} (\pi_\ell(a) = \pi_\ell(b) \wedge \bigwedge_{\ell \in [m, k[} \pi_\ell(b) = \pi_{\ell+1}(a)) \right] \right\}.$$

The idea is that elements of J_*^M must have all k -subsets in them for amalgamation of the $\mathcal{P}^-(n)$ -diagram later and this happens through this representation of the sets.

If M is standard, we have that

$$J_*^M = \left\{ \langle a_\ell | \ell \leq k \rangle \mid \bigwedge_{m \leq k} \langle a_\ell | \ell \neq m \rangle \in J \right\}.$$

Finally, we say that $(\bar{x}, \bar{y}, \bar{z})$ is a **global M -choice** if it is a partial M - $P_{1,1}^M$ -choice. We will sometimes just say M -choice when meaning a global M -choice.

3. Fix a standard M and a M - (J_0, J_1, J_2) -choice $(\bar{x}, \bar{y}, \bar{z})$. Then we let the **correction function** f for M and $(\bar{x}, \bar{y}, \bar{z})$ be the function such that

(a) $\text{Dom}(f)$ is the set of pairs (u, s) such that

(α) $u = \langle a_\ell | \ell \leq k \rangle \in J_2 \subset P_{1,2}^M$,

(β) if $u_m := \langle a_\ell | \ell \leq k, \ell \neq m \rangle$, $u_\ell \in J_0$ for $\ell < k$, $u_k \in J_1 \subset P_{1,1}^M$,

(b) $\text{rng}(f) \subset \mathbb{Z}_2$, and

(c) (recall $x_{u_\ell, s}, y_{u_k, s}, z_{u_k}$ are from the choice)

$$f(u, s) = 0 \Leftrightarrow \langle a_0, \dots, a_k, u_0, \dots, u_k, x_{u_0, s}, \dots, x_{u_{k-1}, s}, y_{u_k, s}, z_{u_k} \rangle \in Q_s^M.$$

The definition makes sense as the λ 's are from the choice.

4. If f is an correction function for some M_I and a M_I -(J_0, J_1, J_2)-choice $(\bar{x}, \bar{y}, \bar{z})$, then f we call f a (I, J_0, J_1, J_2) -correction function.
5. $C(I, J_0, J_1, J_2)$ denotes the set of all (I, J_0, J_1, J_2) -correction functions.

The next three claims are general observations on correction functions and choices.

Claim 3.2 1. If $(\bar{x}, \bar{y}, \bar{z})$ is a global M -choice, $M \models \psi$, and f is the M -correction function for $(\bar{x}, \bar{y}, \bar{z})$, and f is identically zero, then $M \approx M_I$ for some I .

2. If f above is zero on $P_{1,1}^M, P_{1,2}^M$ and $f = f' \upharpoonright J_2 \times S$, then $M \approx M_{P_1, f'}$.

Corollary 3.3 The correction function for $M_{I, f}$ and the canonical M -choice $(\bar{x}, \bar{y}, \bar{z})$ is f .

PROOF Similar to the above: add zeroes to f as in 3.2. □

Claim 3.4 For every $M \in \text{Mod}(\psi)$, there is an M -choice $(\bar{x}, \bar{y}, \bar{z})$.

PROOF Immediate: just construct the tuples. There demands are on each choice separately. There are no demands connecting different choices. □

The next lemma is a crucial step. It shows how to build if possible isomorphisms from arbitrary N in the class K_2 to canonical models $M_{I, f}$.

Lemma 3.5 For every $N \in \text{Mod}(\psi)$ and global N -choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f there are I and \mathbf{h} such that \mathbf{h} is an isomorphism from N onto $M_{I, f}$, ($I = P_0^M = P_0^N$ except if unwanted equations hold - e.g. those failing in some M_I) and

$$\mathbf{h}(x_{u, s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \quad \mathbf{h}(y_{u, s}) = (\mathbf{h}(u), s, 0_{H_I}), \quad \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G).$$

PROOF First, extract the predicates: this provides $I = P_0^M$. Clearly $P_0^M = P_0^N$. Then, by the definition of $M_{I, f}$, we get that

$$\left. \begin{array}{l} x \in P_{1,1} \cup P_{1,2} \\ \ell < k, \pi_\ell(x) = x_\ell \end{array} \right\} \Rightarrow \mathbf{h}(x) = (\mathbf{h}(x_0), \dots, \mathbf{h}(x_{k-1}))$$

and thus the construction of \mathbf{h} on P_2^N , P_3^N , P_4^N and P_5^N should respect the predicates $P_{2,s}^N$ and $P_{3,s}^N$. So we have

$$\left. \begin{array}{l} x \in P_2^N \\ F_2^N(x) = u \in P_{1,1}^N \\ F_4^N(x, h) = x' \end{array} \right\} \Rightarrow F_2^N(\mathbf{h}(x)) = \mathbf{h}(F_2^N(x)) = \mathbf{h}(u)$$

and $F_4^N(\mathbf{h}(x), \mathbf{h}(h)) = \mathbf{h}(x')$.

So,

$$\mathbf{h}(x) = (\mathbf{h}(u), s, -).$$

As the definition of $\mathbf{h} \upharpoonright P_2^N$ is free in the choice of the ‘third coordinate’ element of H this part will only be tied by the correction function. Lastly, we need to check that this definition works together fine with the predicates Q_s . Fix $s \in S$. Finally, checking

$$\begin{array}{c} Q_s^N(a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k z) \\ \Downarrow \\ Q_s^{M_{I,f}}(\mathbf{h}(a_0) \dots \mathbf{h}(a_k) \mathbf{h}(u_0) \dots \mathbf{h}(u_k) \mathbf{h}(x_0) \dots \mathbf{h}(x_{k-1}) \mathbf{h}(y_k) \mathbf{h}(z)). \end{array}$$

amounts to answering the question

$$\begin{array}{c} \mathbb{Z}_2 \models \sum_{\ell < k} i_\ell = h_k(u_0) + g(s) + f(u, s) \\ \Downarrow? \\ \mathbb{Z}_2 \models \sum_{\ell < k} \mathbf{h}(i_\ell) = h_k(\mathbf{h}(u_0)) + g(\mathbf{h}(s)) + f(\mathbf{h}(u), \mathbf{h}(s)) \end{array}$$

Choosing

$$\left\{ \begin{array}{l} \mathbf{h}(x_{u,s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \\ \mathbf{h}(y_{u,s}) = (\mathbf{h}(u), s, 0_H), \\ \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G) \end{array} \right.$$

works for these equations: we are ‘choosing’ 0 on the third coordinates – at the $x_{u,s}$, $y_{u,s}$, $z(u)$ that had already been selected by the choice function.

So, this turns the equation at the choices into

$$\mathbb{Z}_2 \models 0 = \sum_{\ell < k} 0 = 0(\star) + 0(\star) + f(\star).$$

But, since f was a correction function for our choice,

$$f(u, s) = 0 \Leftrightarrow \langle a_0, \dots, a_k, u_0, \dots, u_k, x_{u_0,s}, \dots, x_{u_{k-1},s}, y_{u_k,s}, z_{u_k} \rangle \in Q_s^N,$$

the definition of \mathbf{h} works. □

4 Canonical choices

Definition 4.1 Fix $M = M_{I,f}$, and let $(\bar{x}, \bar{y}, \bar{z})$ be the M -choice given by

$$\begin{aligned} x_{u,s} &= (u, s, 0_{\mathbb{Z}_2}), \\ y_{u,s} &= (u, s, 0_{H_I}), \\ z_u &= (u, 0_G). \end{aligned}$$

This is by definition the **canonical** M -choice².

Here is the crucial lemma.

Lemma 4.2 If M_1 and M_2 are strongly standard, and $(\bar{x}, \bar{y}, \bar{z})_\ell$ is an M_ℓ -choice for M_ℓ ($\ell = 1, 2$), $P_0^{M_1} = P_0^{M_2}$ with correction function f_ℓ for $\ell = 1, 2$ then the following are equivalent:

- (a) there is an isomorphism from M_1 onto M_2 over the identity on $P_0^{M_1} \cup P_1^{M_1}$
- (b)₁ there is an M_2 -choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_1 ,
- (b)₂ there is an M_1 -choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_2 ,
- (c) there are functions g_0, g_1, g_2 (this is to correct the choice of zeros), with
 1. $g_1 : {}^k I \times S \rightarrow \mathbb{Z}_2$ (like the $x_{u,s}$'s above),
 2. $g_2 : {}^k I \times S \rightarrow I_H$ (like the $y_{u,s}$'s above),
 3. $g_3 : {}^{k+1} I \rightarrow G$ (like the z_u 's above),
 4. if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z \rangle$ are like in Definition 2.3 for M_1 , or M_2 then

$$\mathbb{Z}_2 \models \sum_{\ell < k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell < k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

PROOF

- (a) \rightarrow (b)₁ Recall that $M_1 \upharpoonright \tau^- = M_2 \upharpoonright \tau^-$, so M_1 and M_2 have the same universes. Fix $F : M_1 \xrightarrow{\sim} P_0^{M_1} \cup P_1^{M_1} M_2$. We have, since f_1 is a correction function for M for the choice $(\bar{x}, \bar{y}, \bar{z})_1$, that

$$f_1(u, s) = 0 \Leftrightarrow \langle a_0 \dots a_k u_0 \dots u_k x_{u_0 s}^1 \dots x_{u_{k-1} s}^1 y_{u_k s}^1 z_{u_k}^1 \rangle \in Q_s^{M_1}.$$

²So the choices act like ‘variations’ on the group structure of G, H and \mathbb{Z}_2 .

But the right hand side holds iff

$$\langle a_0 \dots a_k u_0 \dots u_k F(x_{u_0 s}^1) \dots F(x_{u_{k-1} s}^1) F(y_{u_k s}^1) F(z_{u_k}^1) \rangle \in Q_s^{M_2},$$

since F is an isomorphism fixing $P_0^{M_1} \cup P_1^{M_1}$, and $a_0, \dots, a_k \in P_0^{M_1}$. This gives us the M_2 -choice for which f_1 is a correction function: given $u_\ell \subset u$, $u_\ell \in {}^k I$, $u \in {}^{k+1} I$, let $x'_{u_\ell, s} = F(x_{u_\ell, s}^1)$, $y'_{u, s} = F(y_{u_k, s}^1)$, $z'_{u_k} = F(z_{u_k}^1)$.

(a) \rightarrow (b)₂ Same.

(b)_ℓ \rightarrow (c) ($\ell = 1, 2$) The point of (c) is that we may find concrete representations g_1, g_2, g_3 , that act *independently from M or N* as ‘corrected choice functions’ for the zeros for f_1 and f_2 . So, suppose we have a M_2 -choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f_1 . Then for any $u \in P_0^{M_2}$ and any $s \in S$, if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z \rangle$ are like in Definition 2.3

$$\langle a_0 \dots a_k u_0 \dots u_k x_{u_0 s} \dots x_{u_{k-1} s} y_{u_k s} z_{u_k} \rangle \in Q_s^{M_2}$$

\Updownarrow

$$f_1(u, s) = 0.$$

But since f_1 is also a correction function for the M_1 -choice $(\bar{x}, \bar{y}, \bar{z})_1$,

$$f_1(u, s) = 0$$

\Updownarrow

$$\langle a_0 \dots a_k u_0 \dots u_k x_{u_0 s}^1 \dots x_{u_{k-1} s}^1 y_{u_k s}^1 z_{u_k}^1 \rangle \in Q_s^{M_1}.$$

So, we have both $\mathbb{Z} \models \sum_{\ell < k} i_\ell = h_k(u_0) + g(s)$ and $\mathbb{Z} \models \sum_{\ell < k} i_\ell^1 = h_k^1(u_0) + g^1(s)$, so setting

$$g_1(u_\ell, s) = i_\ell^1, \quad g_2(u_k, s) = h_k^1, \quad g_3(u) = g^1$$

yields

$$\mathbb{Z}_2 \models \sum_{\ell < k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell < k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s).$$

Since f_1 does this for all possible $k + 1$ -tuples, we have all the compability we need.

- (c) \rightarrow (a) If the predicates are the same modulo g_1, g_2 and g_3 then obtaining (a) becomes a matter of building $F : M_1 \xrightarrow{\cong} P_0^{M_1} \cup P_1^{M_1} M_2$. Clearly we can start by $F \upharpoonright P_0^{M_1} = id$, and then extend its definition to all the other portions of the model. The only strong restriction to the extension of this to the whole model is given by the relations $Q_s^{M_1}$ and $Q_s^{M_2}$.

□

Remark 4.3 1. We shall use ‘simple’ versions of $\langle g_1, g_2, g_3 \rangle$, usually to prove isomorphism (two of them zero).

2. The number of isomorphism types count has some similarity to $Ext(G, \mathbb{Z})$, in particular to the work of Shelah and Väisänen in [SV]. Here $I(\lambda, \psi)$ is counted by the group of correction functions, derived from some g_1, g_2, g_3 :

$$I(\lambda, \psi) = \left\{ f \in C(I, J_0, J_1, J_2) \mid f(u, s) = \sum_{\ell < k} g_1(u_\ell, s) - g_2(u_0, s) - g_3(u) \right\}.$$

3. In the isomorphism proof, we will use the regularity of the filter: we will put together λ demands.

The next lemma is the first step in the categoricity proof. It provides conditions for extending partial M -choices to global M -choices for combination (λ, m) , $m < k$.

Lemma 4.4 (Extension property for W of size $m < k$, $|P_0^M| \leq \lambda$.)

Assume $m < k$, $M \models \psi$, M is strongly standard, $|P_0^M| \leq \lambda$, $W \subset P_0^M$, $W = \{b_\ell \mid \ell < m\}$ with no repetition, $J = \{u \in P_{1,1}^M \mid W \not\subset u\}$ (note that $u \in [P_0^M]^k$, as M is standard), $(\bar{x}, \bar{y}, \bar{z})$ is an M - J -choice with correction function f_0 , identically zero. Then, we can extend $(\bar{x}, \bar{y}, \bar{z})$ to an M -choice with correction function identically zero.

PROOF

Part A: Without loss of generality, by 2.7, since M is strongly standard, $I = P_0^M$. Let $\langle \bar{a}^\alpha \mid \alpha < \beta^* \rangle$ list $P_{1,1}^M$ with $\langle \bar{a}^\alpha \mid \alpha < \alpha^* \rangle$ listing J (we have also used u for naming these \bar{a}^α 's). Let $\langle \bar{b}^\gamma \mid \gamma < \gamma^* \rangle$ list $\{\bar{a} \in {}^{k+1}I \mid \bar{a} \text{ with no repetition and } W \subset \text{rng}(\bar{a})\}$ and $\gamma^* < \lambda^+$. Let, for $\alpha < \alpha^*$,

$$x_{\bar{a}^\alpha, s} = (\bar{a}^\alpha, s, i_{\alpha, s}) \in (\mathbb{Z}_2)_{\bar{a}^\alpha, s}, i_{\alpha, s} \in \mathbb{Z}_2,$$

$$y_{\bar{a}^\alpha, s} = (\bar{a}^\alpha, s, h_{\alpha, s}) \in (H)_{\bar{a}^\alpha, s}, h_{\alpha, s} \in H_I,$$

$$z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g), g \in G.$$

Our hypothesis is then that we have choice functions for all $u \in P_{1,1}^M$ such that $u \not\supset W$. We use a zero correction function - as we don't know yet how to take care of $u \supset W$.

We will now choose $x_{\bar{a}^\alpha, s} = (\bar{a}^\alpha, s, i_{\alpha, s})$, $y_{\bar{a}^\alpha, s} = (\bar{a}^\alpha, s, h_{\alpha, s})$, $z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g)$ for $\alpha^* \leq \alpha < \beta^*$ and appropriate γ .

Without loss of generality, $\beta^* \leq \alpha^* + \lambda$, $\gamma^* \leq \lambda$. (Remember $S = [\lambda]^{<\aleph_0}$.)

Part B: First, we choose $i_{\alpha, s} = 0_{\mathbb{Z}_2}$ for $\alpha^* \leq \alpha < \beta^*$, $s \in S$.

Second, we are now at the level of consistently 'choosing h '. We try to choose $h_{\alpha, s}$ for $\alpha^* \leq \alpha < \beta^*$ and $s \in S$ such that

(*) if $\gamma \in s \subset \lambda$, $\bar{b}^\gamma = \langle b_\ell^\gamma \mid \ell \leq k \rangle$, $u_n^\gamma = \langle b_\ell^\gamma \mid \ell \leq k, \ell \neq n \rangle$, let $\varepsilon(\gamma, n) < \beta^*$ be such that $u_n^\gamma = \bar{a}^{\varepsilon(\gamma, n)}$ then

$$h_{\varepsilon(\gamma, k), s}(\bar{a}^{\varepsilon(\gamma, 0)}) = 0$$

\Updownarrow

$$\langle b_0^\gamma, \dots, b_k^\gamma, u_0^\gamma, \dots, u_k^\gamma, x_{\varepsilon(\gamma, 0)}, \dots, x_{\varepsilon(\gamma, k-1)}, (\bar{a}^{\varepsilon(\gamma, k)}, s, 0_H), (u^\gamma, 0_G) \rangle \in Q_s^M.$$

Note that all the elements in the bottom part are defined.

Let $t(\gamma, s)$ be 0 if the bottom statement is true, 1 otherwise (so we are using \mathbb{Z}_2 to code). This gives $|s|$ demands, one for each $\gamma \in s$. The sequence $\langle \bar{a}^{\varepsilon(\gamma, 0)} \mid \gamma \in s \rangle$ is without repetition (see part B). So we have to show that the set of equations in the variable h varying on $H = [{}^k I]^{<\aleph_0}$, considered as a set of characteristic functions is

$$\{h(\bar{a}^{\varepsilon(\gamma, 0)}) = t(\gamma, s) \mid \gamma \in s\}.$$

By the definition, it is solvable by the characteristic function of the subset $\{\bar{a}^{\varepsilon(\gamma, 0)} \mid \gamma \in s\}$ of ${}^k I$.

The decisions are done for each s separately, also fixing s we can deal with one $\alpha \in [\alpha^*, \beta^*] \setminus \{\beta^*\}$ e.g. choosing $h_{\alpha, s}$ we have to consider only $\gamma < \gamma^*$ such that $\{\varepsilon(\gamma, \ell) \mid \ell < k\} \subset s$; there are here only finitely many γ 's, and if $\gamma_1 \neq \gamma_2 \in s$ (and $\varepsilon(\gamma_1, k) = \alpha = \varepsilon(\gamma_2, k)$) necessarily $\varepsilon(\gamma_1, 0) \neq \varepsilon(\gamma_2, 0)$ (as \bar{a}^γ is reconstructible from α and $\varepsilon(\gamma_1, 0)$), i.e. if equality holds then $\bar{b}^{\gamma_1} = \bar{b}^{\gamma_2}$) and by the choice of H we can find $h_{\varepsilon(\gamma, k), s}$.

Part C: We now ‘glue’ the choices, for fixed γ . For each $\bar{b} \in {}^{k+1}I$, $\bar{b} = \bar{b}^\gamma$ for some $\gamma < \gamma^*$, so

$$S_\gamma^* = \left\{ s \in S \mid M \models Q_s(b_0^\gamma, \dots, b_k^\gamma, \bar{a}^{\epsilon(\gamma,0)}, \dots, \bar{a}^{\epsilon(\gamma,k)}, x_{u_0^\gamma, s}, \dots, x_{u_{k-1}^\gamma, s}, y_{u_k^\gamma, s}, (u^\gamma, 0_S)) \right\}$$

belongs to \mathfrak{D} (by the regularity of \mathfrak{D}).

Next choose $z_{\bar{b}_1} = (\bar{b}, g)$ by

$$g(s) = \begin{cases} 0 & \text{if } s \in S_\gamma^* \\ 1 & \text{if } s \notin S_\gamma^* \end{cases}$$

Now then, with these x , y and z , the equation holds.

□_{4.4}

We now prove the general extension property.

Lemma 4.5 (Full extension)

Let $M \models \psi$ be strongly canonical, $J_1 \subset J_2 \subset P_0^M$, with $|J_2| < \lambda^{+k-1}$ and $(\bar{x}, \bar{y}, \bar{z})$ an M - J_1 -choice with correction function identically zero. Then $(\bar{x}, \bar{y}, \bar{z})$ can be extended to an M - J_2 -choice with correction function identically zero.

PROOF Without loss of generality, $J_2 = J_1 \cup \{b\}$. If J_1 has size $\leq \lambda$, this is lemma 4.4. Now suppose $|J_1| = \lambda^{+m_1}$ (for $m_1 < k$), so enumerate J_1 as $\langle a_\beta \mid \beta < \lambda^{+m_1} \rangle$. Let $J_1^\alpha = \{a_\beta \mid \beta < \alpha\}$, and let $(\bar{x}, \bar{y}, \bar{z})_\alpha$ be the restriction of $(\bar{x}, \bar{y}, \bar{z})$ to an M - J_1^α -choice. We define by induction M - J_1^α choices with correction function identically zero $(\bar{x}, \bar{y}, \bar{z})'_\alpha \supset (\bar{x}, \bar{y}, \bar{z})_\alpha$. Use lemma 4.9 for $m_2 = 2$ to extend $(\bar{x}, \bar{y}, \bar{z})'_\alpha \cup (\bar{x}, \bar{y}, \bar{z})_{\alpha+1}$ to an M - $J_1^{\alpha+1} \cup \{b\}$ -choice with correction function identically zero. At limits take unions; finally,

$$\left(\bigcup_{\alpha < \lambda^{+m_1}} \bar{x}'_\alpha, \bigcup_{\alpha < \lambda^{+m_1}} \bar{y}'_\alpha, \bigcup_{\alpha < \lambda^{+m_1}} \bar{z}'_\alpha \right)$$

turns out to be an M - J_2 -solution extending $(\bar{x}, \bar{y}, \bar{z})$. □

Claim 4.6 In 4.4, we can allow $|P_0^M| \leq \lambda^{+m_1}$ if $W = \{b_\ell \mid \ell < m_2\}$ and $m_1 + m_2 < k$.

PROOF We prove this by induction on m_1 . The proof is quite parallel to some of the proofs in [She83a] and to [HS90]. This part has few changes. We include versions of those proofs adapted to our context.

For $m_1 = 0$, this was done in 4.4. Suppose it holds for $m_1 (< k)$, and m_2 is such that $m_1 + m_2 < k$. Consider $W \subset P_0^M$,

$$W = \{b_\ell \mid \ell < m_2\},$$

$(m_1 + 1) + m_2 < k$, $J = \{u \in P_1^M \mid W \not\subset u\}$, and let $(\bar{x}, \bar{y}, \bar{z})$ be an M - J -choice with correction function identically 0.

Suppose, for $M \models \psi$ strongly standard, $A_\emptyset \subset P_0^M$, a_0, \dots, a_{m_2-1} different elements of $P_0^M \setminus A_\emptyset$,

Definition 4.7 $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s \mid s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ^{+m_1} - $\mathcal{P}^-(m_2)$ -system of choices iff

1. $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s = A_\emptyset \cup \{a_0, \dots, a_{m_2-1}\}$, $|A_\emptyset| \leq \lambda^{+m_1}$, $A_s = A_\emptyset \cup \{a_t \mid t \in s\}$.
2. $(\bar{x}, \bar{y}, \bar{z})_s$ is a M - A_s -choice, $\forall s \in \mathcal{P}^-(m_2)$.
3. For every $s, t \in \mathcal{P}^-(m_2)$, $s \subset t \Rightarrow (\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$ ³.

Lemma 4.8 If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s \mid s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ - $\mathcal{P}^-(m_2)$ -system with $m_2 < k$ then there is an M - $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$ -choice $(\bar{x}, \bar{y}, \bar{z})$ extending all the $(\bar{x}, \bar{y}, \bar{z})_s$, for $s \in \mathcal{P}^-(m_2)$.

PROOF If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s \mid s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ - $\mathcal{P}^-(m_2)$ -system with $m_2 < k$, if

$$u \in \left[\bigcup_{s \in \mathcal{P}^-(m_2)} A_s \right]^k \setminus \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k,$$

then $\{a_0 \dots a_{m_2-1}\} \subset u$. As $m_2 < k$, there must be some $b \in u \setminus \{a_0 \dots a_{m_2-1}\} \subset u$. Now, if $c \in \bigcup_{s \in \mathcal{P}^-(m_2)} A_s \setminus u$ then

$$(u \setminus \{b\}) \cup \{c\} \notin \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$$

hence if $u \subset v$ where $v \in \left[\bigcup_{s \in \mathcal{P}^-(m_2)} A_s \right]^{k+1}$ then there must exist $u' \subset v$, $|u'| = k$, $u \neq u'$ such that $u' \notin \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$. □_{4.8}

Lemma 4.9 If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s \mid s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ^{+m_1} - $\mathcal{P}^-(m_2)$ -system of choices with $m_1 + m_2 < k$ then there is a $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$ -choice $(\bar{x}, \bar{y}, \bar{z})$ such that $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})$ for every $s \in \mathcal{P}^-(m_2)$.

PROOF Induct on m_1 . For $m_1 = 0$, this is lemma 4.8. For $m_1 > 0$, suppose $A_s = A_\emptyset \cup \{b_t \mid t \in s\}$. Enumerate A_\emptyset , $\langle a_\beta \mid \beta < \lambda^{+m_1} \rangle$ and let $A_\emptyset^\alpha = \{a_\beta \mid \beta < \alpha\}$. Now let $A_s^\alpha = A_\emptyset^\alpha \cup \{b_t \mid t \in s\}$ for every $s \in \mathcal{P}^-(m_2)$ and $(\bar{x}, \bar{y}, \bar{z})_s^\alpha$ the restriction of $(\bar{x}, \bar{y}, \bar{z})_s$ to an M - A_s^α -choice. The point is to get $(\bar{x}, \bar{y}, \bar{z})_\alpha$ (increasing with α for $\alpha < \lambda^{+m_1}$ such that $(\bar{x}, \bar{y}, \bar{z})_\alpha$ is an M - $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s^\alpha$ -choice, with $(\bar{x}, \bar{y}, \bar{z})_\alpha \supset (\bar{x}, \bar{y}, \bar{z})_s^\alpha$ (the obvious meaning, again) for every $s \in \mathcal{P}^-(m_2)$. With this, taking $(\bar{x}, \bar{y}, \bar{z}) = \bigcup_{\alpha < \lambda^{+m_1}} (\bar{x}, \bar{y}, \bar{z})_\alpha$ we are done. [Why is the construction possible? At limits, just take unions. At successors, use the induction hypothesis.] □_{4.9}

³here, of course, we are abusing notation - by $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$ we mean $\bar{x}_s \subset \bar{x}_t$, $\bar{y}_s \subset \bar{y}_t$ and $\bar{z}_s \subset \bar{z}_t$.

Theorem 4.10 *If $M \models \psi$ is strongly canonical and $|M| < \lambda^{+k}$ then there is an M -choice with correction function identically zero.*

PROOF Use the previous lemmas and induction .

Conclusion 4.11 (Categoricity and amalgamation up to $\lambda^{+(k-1)}$)

1. For $m < k$, $\text{Mod}(\psi)$ has a unique model M , $|P_0^M| = \lambda^{+m}$.
2. For $m < k - 2$, if $2^\lambda \leq \lambda^{+m}$, then $\text{Mod}(\psi)$ has amalgamation in λ^{+m} .
3. If $m < k$, $\lambda^{+m} > 2^\lambda$, then ψ is categorical in λ^{+m} .

PROOF This is just a summary of the previous arguments. Notice, however, that we require $2^\lambda < \lambda^{+m}$ or $2^\lambda \leq \lambda^{+m}$ for our conclusions. This is due to the fact that our models are large: they contain copies of G , so they have size at least 2^λ . \square

5 ψ is not categorical above $\beth_{k+1}(\lambda)^+$

We proved in 4.11 that ψ is categorical in λ^{+m} if $m < k$ and $2^\lambda < \lambda^{+m}$. We now prove it is not totally categorical, and we will actually get the maximal number of models.

As we will see, the control of isomorphism here is relatively easier than lifting isomorphism in the previous section.

The following fact is a consequence of 4.2. We will rely on a variant of it.

Fact 5.1 *If M_{1,f_1} and M_{2,f_2} are models of ψ , and $h : I_1 \rightarrow I_2$ is one-to-one and onto, then there is an isomorphism $h^+ : M_{1,f_1} \rightarrow M_{2,f_2}$ extending h iff there are functions g_0, g_1, g_2 (this is to correct the choice of zeros), with*

1. $g_1 : {}^k I \times S \rightarrow \mathbb{Z}_2$ (like the $x_{u,s}$'s above),
2. $g_2 : {}^k I \times S \rightarrow I_H$ (like the $y_{u,s}$'s above),
3. $g_3 : {}^{k+1} I \rightarrow G$ (like the z_u 's above),
4. if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z \rangle$ are like in Definition 2.3 for M_1 , or M_2 then

$$\mathbb{Z}_2 \models \sum_{\ell < k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell < k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

Claim 5.2 Let $f : [I]^{k+1} \times S \rightarrow \mathbb{Z}_2$ be such that

$$\otimes \quad \{s \in S \mid f_u \neq 0\} \in \mathfrak{D}, \forall u \in [I]^{k+1},$$

for $f_u : S \rightarrow \mathbb{Z}_2$ with $f_u(s) = f(u, s)$.

Then, the following is a sufficient condition for

$$M_{I,f} \not\approx M_I :$$

(*) if $F_1 : {}^k I \rightarrow [I]^{\leq \lambda}$, $F_2 : {}^k I \rightarrow S(\mathbb{Z}_2)$ π a permutation of I , then we can find $t_0, \dots, t_k \in I$ with no repetitions such that

- (α) $t_\ell \notin F_1[\{t_0 \dots t_k\} \setminus \{t_\ell\}]$ if $\ell = k$,
- (β) $f_{\pi\{t_0, \dots, t_k\}} - \sum_{\ell < k} F_2(\{t_0, \dots, t_k\} \setminus \{t_\ell\}) \notin G$

So, by the definition of G ,

$$\{s \in S \mid f_{\pi(t_0, \dots, t_k)}(s) = \sum_{\ell < k} F_2(\{t_0, \dots, t_k\} \setminus \{t_\ell\})(s)\} \notin \mathfrak{D}.$$

Before proving 5.2, we note some facts.

Definition 5.3 $f : [I]^{k+1} \times S \rightarrow \mathbb{Z}_2$ is an I -function iff it satisfies \otimes above.

Fact 5.4 If f_1, f_2 are I -functions, and $f = f_1 - f_2$ (coordinatewise) satisfies (*), then $M_{I,f_1} \not\approx M_{I,f_2}$.

PROOF By 4.2, we can apply the criterion (*) to the model. \square

Fact 5.5 In 5.2, the assumption on f does not entail a loss of generality, as for every f there is a f' as above such that $M_{I,f} \approx M_{I,f'}$.

Notice the role of the permutation π of I in the combinatorics that follows.

PROOF of 5.2. Assume that $(\bar{x}, \bar{y}, \bar{z})$ witnesses $M_{I,f} \approx M_I$, with correction function identically zero. We show that (*) of 5.2 cannot hold, for the following choice of F_1 and F_2 .

Define $F_1 : [I]^k \rightarrow [I]^{\leq \lambda}$ by

$$F_1(u) = \bigcup \{v \in {}^k I \mid \text{for some } s_1 \in S, y_{u, s_1}(v) \neq 0\}.$$

This is well defined, as $F_1(u)$ is a union of $|S|$ finite sets. Also, set

$$F_2(u) = \langle x_{u, s} \mid s \in S \rangle.$$

Let now $t_0, \dots, t_k \in I$ (with no repetitions) satisfy $(\alpha) + (\beta)$. Let as usual $u = \{t_0, \dots, t_k\}$, $u_\ell = u \setminus \{t_\ell\}$. By (α) ,

$$y_{u_k, s}(u_0) = 0.$$

[Just notice that (α) asks that $t_k \notin F_1(u_k) = \bigcup \{v \in {}^k I \mid \text{for some } s_1 \in S, y_{u, s_1}(v) \neq 0\}$, so for all $v \in [I]^k$, if $t_k \in v$, then for all $s_1 \in S$ we have $y_{u_k, s_1}(v) = 0$. In particular, as $t_k \in u_0$, $y_{u_k, s_1}(u_0) = 0$.]

So, by the choice $(\bar{x}, \bar{y}, \bar{z})$ and since we chose our correction function to be identically zero, for each z , we have that

$$\mathbb{Z}_2 \models x_{u_0, s} + x_{u_1, s} + \dots + x_{u_{k-1}, s} - y_{u_k, s} - z_u = f_{\pi(u_0), s} + \dots + f_{\pi(u_{k-1}), s} - f_{u_k, s}(u_0) - f_u(s).$$

But we also have that $f_u(s)$ is not zero (initial assumption) and $z_u(s) = 0$ for the \mathfrak{D} -majority of $s \in S$ (by the definition of G). Also, $y_{u_k, s}(u_0) = 0$, by the choice of the t 's (clause (α)). So,

(*) For the \mathfrak{D} -majority of $s \in S$

$$\sum_{\ell < k} x_{u_\ell, s} = f_{\pi(u)}(s).$$

But this contradicts (β) . □

Remark 5.6 *We can then regard F_2 as*

$$F_2 : {}^k I \rightarrow S(\mathbb{Z}_2)/G.$$

Conclusion 5.7 *For $\mu = \beth_{k+1}(\lambda)^+$, ψ is not categorical.*

This is not optimal (μ is large) but is enough for our main aim. In a possible continuation, we will address this issue.

PROOF We take advantage of the combinatorial reduction from 5.2.

STAGE A: First, let k be even. There is f an I -function (as in 5.2). Now, assume that F_1, F_2 are as in 5.2, and derive a contradiction. first find $E \subset \mu$ club such that

$$\alpha_0 < \dots < \alpha_k \in E \implies \begin{cases} F_1(\alpha_0, \dots, \alpha_{k-1}) \subset \alpha_k, \\ \pi(\alpha_0), \dots, \pi(\alpha_{k-1}) < \alpha_k. \end{cases}$$

Apply then Erdős-Rado to F_2 in order to get $\alpha_0 < \dots < \alpha_k$ in E with $u = \{\alpha_0, \dots, \alpha_k\}$, $u_\ell = u \setminus \{\alpha_\ell\}$, $\langle F_2(u_\ell) \mid \ell < k \rangle$ constant.

But then

$$\sum_{\ell < k} F_2(u_\ell) = 0,$$

as our group is of order 2 (and k was chosen to be an even number).

STAGE B: More generally, choose μ such that

$$\otimes_1 \mu \rightarrow (\omega)_{2^\lambda}^k,$$

$$\otimes_2 \mu \not\rightarrow (\omega)_{2^\lambda}^{k+1},$$

$$\otimes_3 \mu \text{ regular.}$$

Then, use f as in 5.2 exemplifying \otimes_2 . Looking at f as $(u \mapsto f_u/G \in ({}^S(\mathbb{Z}_2)/G))$, toward contradiction, if F_1, F_2 are as in 5.2, we let $E \subset \mu$ club as above, $\alpha_0, \dots, \alpha_n, \dots$ exemplifying $\mu \rightarrow (\omega)_{2^\lambda}^k$ for the coloring F_2 . So $F_2 \upharpoonright [\{\alpha_0, \dots, \alpha_n, \dots\}]^k$ is constant, say for increasing k -tuples from E , hence also the coloring by c by the argument as above, contradicting its choice.

STAGE C: For larger cardinals this obviously works, as the criterion is monotonic. \square

Remark 5.8 *Here are some of the main differences between the structure of this proof and that of [HS90]:*

1. *The use of the filter \mathfrak{D} - it is not needed there.*
2. *The way the group itself is used is slightly different at the end of the proof.*

Remark 5.9 *We get even many models (maximal number) but later...*

PROOF By categoricity, or directly back and forth.

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